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2006 J. Phys. A: Math. Gen. 39 2797

(http://iopscience.iop.org/0305-4470/39/11/015)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 2797–2841 doi:10.1088/0305-4470/39/11/015

Superstring theory on pp waves with ADE geometries

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Received 13 September 2005
Published 1 March 2006
Online at stacks.iop.org/JPhysA/39/2797

Abstract

We study the BMN correspondence between certain Penrose limits of type IIB superstrings on pp-wave orbifolds with ADE geometries and the set of four-dimensional $\mathcal{N} = 2$ superconformal field theories constructed as quiver gauge models classified by finite ADE Lie algebras and affine $\widehat{\text{ADE}}$ Kac–Moody algebras. These models have 16 preserved supercharges and are based on systems of D3-branes and wrapped D5- and D7-branes. We derive explicitly the metrics of these pp-wave orbifolds and show that the BMN extension requires, in addition to D5–D5 open strings in bi-fundamental representations, D5–D7 open strings involving orientifolds with $Sp(N)$ gauge symmetry. We also give the correspondence rule between leading string states and gauge-invariant operators in the $\mathcal{N} = 2$ quiver gauge models.

PACS numbers: 11.25.−w, 02.40.−k, 02.20.Sv

1. Introduction

The AdS/CFT correspondence relates the spectrum of type IIB superstring theory on $\text{AdS}_5 \times S^5$ to the spectrum of single-trace operators in the large-N limit of $\mathcal{N} = 4U(N)$ SYM_4 gauge theory on the boundary of AdS_5 [1–9]. The proposal by Berenstein, Maldacena and Nastase (BMN) predicts even more [10], as it offers one to derive the spectrum of the superstring theory from the gauge-theory point of view, not only on flat space as in the AdS/CFT correspondence, but also on a plane-wave background obtained as a Penrose limit of $\text{AdS}_5 \times S^5$. The type IIB superstring states on plane-polarized (pp) waves preserving all 32 supercharges are thus related to gauge-invariant conformal operators of $\mathcal{N} = 4$ CFT_4 in the worldvolume of N D3-branes.

This correspondence has already been studied extensively in the literature, and we refer to [11–21] for more details. The BMN proposal has recently been extended successfully to models preserving 16 supercharges [22–26]. These ideas are pursued further in the present work, where we present an explicit analysis of the correspondence between type IIB superstring theory on pp-wave orbifolds and the dual $\mathcal{N} = 2$ quiver CFT$_4$s classified by affine $\widehat{\mathrm{ADE}}$ Kac–Moody algebras or by finite ADE Lie algebras with a specific open-string sector.

As already indicated, the BMN proposal offers a prescription for relating chiral primary operators from the conformal model to states in the superstring theory. Let Δ denote the conformal weight of the complex field Z, i.e., its canonical dimension in CFT$_4$, and let J be the charge under an $SO(2)$ R-symmetry rotating two of the six scalars of the $\mathcal{N} = 4$ multiplet. For large N, a gauge-invariant complex field operator like $\mathrm{Tr}[Z^J]$ with charge $J \sim \sqrt{N}$ and $\Delta - J = 0$, for example, is thus proposed to be associated with the vacuum string state $|0, p^+\rangle_{\mathrm{lc}}$ in the light-cone gauge with large momentum p^+. In the string model, the parameters Δ and J are the eigenvalues of the energy operator $\mathrm{i}\partial/\partial t$ and the angular-momentum operator $-\mathrm{i}\partial/\partial\psi$ (with respect to the great circle of S^5), respectively, in which case $\Delta - J$ corresponds to the energy of a light-cone state. The proposed correspondence can also be worked out explicitly for small (integer) values of $\Delta - J = n$ and in principle for the full tower of string states $|n, p^+\rangle$ and conformal operators $\mathcal{A}_{\Delta,J} \sim \mathrm{Tr}[\mathrm{A}_{\Delta,J} Z^J]$.

The BMN proposal in its original form concerns a scenario with 32 preserved supercharges and has recently been extended to cases preserving 16 conformal supercharges. An objective of the present work is to indicate how this may be extended further to models preserving an integral fraction of the original 32 conformal supercharges. Our method is developed along the lines of [27, 28] and the results are obtained by soft truncations of $\mathcal{N} = 4U(N)$ SYM$_4$ gauge theory. In view of our construction, the studies of [22, 24] merely become particular cases preserving 16 conformal supercharges. The analysis in [22] corresponds to an orbifold group $\Gamma \subset SU(2)$, while the work [24] deals with closely related models with a symplectic gauge group. As in the case of [22], the orbifold groups Γ of interest to our analysis are subgroups of an $SU(2)$ subgroup of the $SU(4) \sim SO(6)R$-symmetry associated with the space $\mathbb{R}^6$ transverse to the aforementioned D3-branes containing the CFT$_4$ in their worldvolume. A substantial part of the present work is therefore devoted to the case with 16 preserved supercharges. The following naive picture summarizes the paths we will follow to achieve our goals:

$$\boxed{\begin{array}{lcl}
\text{Type IIB on } \mathrm{AdS}_5 \times S^5 \text{ pp waves} & \Longleftrightarrow & \mathcal{N} = 4\ \ U(N)\ \ \mathrm{SYM}_4/\mathrm{CFT}_4 \\
\textit{Orbifolding by } \Gamma \subset SU(2),\ \ |\Gamma| = k & \Downarrow\Downarrow & \textit{Orbifolding by } \Gamma \subset SU(2),\ |\Gamma| = k \\
\text{Type IIB on pp-wave orbifolds} & & U(kN)\ \mathrm{SCFT}_4 \text{ on D-branes} \\
\text{away from the fixed point} & & \text{away from the fixed point} \\
\textit{Deformation of orbifold singularity} & \Downarrow\Downarrow & \textit{Deformation of orbifold singularity} \\
\text{Type IIB on deformed pp-wave orbifolds} & \Longleftrightarrow & \mathcal{N} = 2\ \left[\otimes_{i=1}^{k} U(N_i)\right]\ \mathrm{CFT}_4
\end{array}} \tag{1.1}$$

Details on the various steps will be given in the main body of the present paper. At this point, we merely wish to point out that in the presence of Chan–Paton factors, the brane engineering of supersymmetric CFT$_4$ will involve not only D3-branes and wrapped D5-branes, but also D7-branes wrapped around 4-cycles with large volume. This property makes the brane engineering of the models more subtle than described in [22]. It also offers clues to the case $\Gamma \subset SU(3)$ (and even $\Gamma \subset SU(4)$) and indicates how one may descend to models with eight conformal supercharges. Our explicit analysis is partly motivated by this last issue, especially in light of the recent developments on $\mathcal{N} = 1$ quiver gauge theories inheriting basic properties of $\mathcal{N} = 2$ quiver SYM$_4$ theories [27]. We intend to explore these ideas further in order to extend the above picture to dual models preserving eight conformal supercharges.

The remaining part of this paper is organized as follows. In section 2, we review results on the BMN proposal and on pp-wave geometries preserving 16 supercharges, and give a brief outline of our construction. In section 3, we study type IIB superstrings on pp waves with ADE orbifold structures, with particular emphasis on the various metrics. In section 4, we discuss $\mathcal{N} = 2$ CFT$_4$s classified by finite ADE and affine $\widehat{\text{ADE}}$ Dynkin diagrams. Details on the non-Abelian cases are deferred to appendix A. Since most known results are on the affine case, we take this opportunity to supplement the literature with explicit discussions on the finite cases. In section 5, we give the brane interpretation of these models. In section 6, we discuss the correspondence between type IIB superstring on pp-wave orbifolds and $\mathcal{N} = 2$ quiver CFT$_4$s. Section 7 is devoted to some concluding remarks. An interpretation of conformal invariance in terms of toric geometry is given in appendix B.

2. Strings on pp waves

2.1. The BMN proposal

We first recall the usual metric of the anti-de Sitter space AdS$_5 \times S^5$ [10],

$$\mathrm{d}s^2 = R^2\big[-\cosh^2(\rho)\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2(\rho)\,\mathrm{d}\Omega_3^2 + \cos^2(\theta)\,\mathrm{d}\psi^2 + \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\Omega_3'^2\big], \tag{2.1}$$

and consider the coordinate transformation

$$x^+ = \frac{t+\psi}{2}, \qquad x^- = R^2\frac{t-\psi}{2}, \qquad x = R\rho, \qquad y = R\theta. \tag{2.2}$$

In the Penrose limit where

$$R \to \infty, \tag{2.3}$$

the geometry is altered, and the new metric may be written as

$$\mathrm{d}s^2 = -4\,\mathrm{d}x^-\,\mathrm{d}x^+ - (\mathbf{x}^2 + \mathbf{y}^2)(\mathrm{d}x^+)^2 + \mathrm{d}\mathbf{x}^2 + \mathrm{d}\mathbf{y}^2. \tag{2.4}$$

Here $\mathbf{x} = (x^1, x^2, x^3, x^4)$ and $\mathbf{y} = (y^1, y^2, y^3, y^4)$ each parameterizes $\mathbb{R}^4$, and $|\mathbf{x}| = x$, $|\mathbf{y}| = y$. After the simple rescaling $x^- \to x^-/\mu$, $x^+ \to \mu x^+$, and by combining the two four-dimensional coordinates into a single one, $\mathbf{w} = (w^1, \ldots, w^8)$, parameterizing $\mathbb{R}^8$, this metric is recognized as a plane-wave metric:

$$\mathrm{d}s^2 = -4\,\mathrm{d}x^-\mathrm{d}x^+ - \mu^2\mathbf{w}^2(\mathrm{d}x^+)^2 + \mathrm{d}\mathbf{w}^2. \tag{2.5}$$

In the following, though, we shall mainly use the coordinates $(\mathbf{x}, \mathbf{y})$. The mass parameter, μ, is related to the 5-form field strength

$$F = \mu\,\mathrm{d}x^+ \wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 \wedge \mathrm{d}y^4). \tag{2.6}$$

Following [10], one may consider the energy and angular-momentum generators as given by $E = \mathrm{i}\partial_t$ and $J = -\mathrm{i}\partial_\psi$, respectively, in terms of the original AdS$_5 \times S^5$ coordinates (2.1). One then finds the following expressions for the light-cone momenta:

$$2p^- = \mathrm{i}\partial_{x^+} = \mathrm{i}(\partial_t + \partial_\psi) = \Delta - J, \qquad 2p^+ = \mathrm{i}\partial_{x^-} = \frac{\mathrm{i}}{R^2}(\partial_t - \partial_\psi) = \frac{\Delta + J}{R^2}, \tag{2.7}$$

where the BPS condition reads $\Delta \geqslant |J|$. The Penrose limit (2.3) of the string model now corresponds to the large-N limit of the dual $\mathcal{N} = 4U(N)$ gauge theory with focus on operators for which $J \sim N^{1/2}$, while $\Delta - J$ and the gauge coupling remain fixed. With these properties in mind, BMN [10] have worked out the correspondence between the spectrum of string states and the spectrum of gauge-model operators. For the minimum value $\Delta - J = 0$, for instance, the correspondence is between the single-trace operator $\mathrm{Tr}[Z^J]$, with Z a complex scalar, and the vacuum state in light-cone gauge $|0, p^+\rangle_{\mathrm{lc}}$. At the next level, where $\Delta - J = 1$, the gauge-invariant field operators of the form $\sum_{l=0}^{J} \mathrm{Tr}[Z^l \phi^r Z^{J-l}]$ and $\sum_{l=0}^{J} \mathrm{Tr}\big[Z^l \psi^b_{\frac{1}{2}} Z^{J-l}\big]$ are associated with $a_0^{\dagger k}|0, p^+\rangle_{\mathrm{lc}}$ and $S_0^{\dagger b}|0, p^+\rangle_{\mathrm{lc}}$, where ϕ^r, $r = 1, 2, 3, 4$, are neutral bosonic scalars with respect to J, while $\psi^b_{\frac{1}{2}}$, $b = 1, \ldots, 8$, are fermionic fields, and $a_0^{\dagger i}$ and $S_0^{\dagger b}$ with $i, b = 1, \ldots, 8$ are bosonic and fermionic zero-momentum oscillators, respectively. For $\Delta - J = 2$, operators of the form $\sum_{l=0}^{J} \mathrm{Tr}\big[\phi^r Z^l \psi^b_{\frac{1}{2}} Z^{J-l}\big]$ are associated with states of the form $a_0^{\dagger r} S_0^{\dagger b}|0, p^+\rangle_{\mathrm{lc}}$. The general correspondence rule is outlined in [10].

Shortly after the BMN proposal appeared, extensions to plane-wave backgrounds preserving 16 spacetime supercharges were considered. These models are based on orbifolds or orientifolds. Since similar constructions are at the core of the present work, a brief summary now follows.

2.1.1. Orbifolds. Recently, Kim, Pankiewicz, Rey and Theisen (KPRT) extended the BMN proposal to type IIB superstrings propagating on pp waves with one of the $\mathbb{R}^4$ factors in $\mathbb{R}^8 \sim \mathbb{R}^4 \times \mathbb{R}^4$ replaced by an $\mathbb{R}^4/\mathbb{Z}_k$ orbifold [22]. It is convenient to describe the orbifold structure in terms of complex coordinates, replacing $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4$ by $(\mathbf{x}, z_1, z_2) \in \mathbb{R}^4 \times \mathbb{C}^2$. The orbifold action is then defined as

$$\mathbb{Z}_k : \quad (z_1, z_2) \mapsto (\omega z_1, \overline{\omega} z_2), \qquad \omega = \exp\left(\frac{2\pi \mathrm{i}}{k}\right), \tag{2.8}$$

where $z_1 = (y^1 + \mathrm{i}y^2)$ and $z_2 = (y^3 + \mathrm{i}y^4)$. Due to the orbifold structure, the string theory will have twisted sectors indexed by the charge, q, of the orbifold group. Physical string states are obtained by applying the bosonic and fermionic creation operators to the light-cone vacuum $|0, p^+\rangle_q$ of each twisted sector labelled by $q = 1, \ldots, k$.

According to [22], the associated superstring theory is described by the large-N limit of $\mathcal{N} = 2\ [U(N)]^k$ quiver gauge theory with fixed gauge coupling and k hypermultiplets in bi-fundamental representations but without fundamental matter. They have also proposed a precise map between the gauge-theory operators and the string states for both untwisted and twisted sectors. The orbifold structure presented here reduces the 32 supercharges of the BMN pp-wave background to 16 supercharges.

As in the BMN correspondence, we here focus on states with conformal weight Δ and $U(1)$ charge $J \sim N^{1/2}$ whose difference $\Delta - J$ remains finite in the large-N limit. Due to the $\mathbb{Z}_k$ orbifolding, some of the fields get promoted to $kN \times kN$ matrices. This applies to the gauge field A_μ, the complex scalar Z associated with the coordinates x^3 and x^4, and the complex scalars $\phi^m = (\phi^1, \phi^2)$ associated with the coordinates $\mathbf{y}$ as well as their superpartners χ and ξ, which are promoted to $\mathcal{A}_\mu$, $\mathcal{Z}$, Φ^m, Λ and Ξ, respectively. They satisfy the following conditions:

$$\begin{aligned} &S\mathcal{A}_\mu S^{-1} = \mathcal{A}_\mu, \qquad && S\mathcal{Z}S^{-1} = \mathcal{Z}, \qquad && S\Lambda S^{-1} = \Lambda, \\ &S\Phi^m S^{-1} = \omega\Phi^m, \qquad && S\Xi S^{-1} = \omega\Xi, && \end{aligned} \tag{2.9}$$

where $S = \mathrm{diag}(1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-k+1})$. It was then proposed that gauge-invariant field operators satisfying $\Delta - J = 0$ are associated with the vacuum in the qth twisted sector

$$\mathrm{Tr}[S^q \mathcal{Z}^J] \quad \leftrightarrow \quad |0, p^+\rangle_q, \tag{2.10}$$

while for $\Delta - J = 1$ we have

$$\begin{aligned} \mathrm{Tr}[S^q \mathcal{Z}^J \mathcal{D}_\mu \mathcal{Z}] \quad &\leftrightarrow \quad a_0^{\dagger\mu}|0, p^+\rangle_q, \\ \mathrm{Tr}\big[S^q \mathcal{Z}^J \chi_{\frac{1}{2}}\big] \quad &\leftrightarrow \quad \chi_0^\dagger|0, p^+\rangle_q, \\ \mathrm{Tr}\big[S^q \mathcal{Z}^J \overline{\chi}_{\frac{1}{2}}\big] \quad &\leftrightarrow \quad \overline{\chi}_0^\dagger|0, p^+\rangle_q, \end{aligned} \tag{2.11}$$

where $\mathcal{D}_\mu \mathcal{Z} = \partial_\mu \mathcal{Z} + [\mathcal{A}_\mu, \mathcal{Z}]$. Operators corresponding to higher string states may also be constructed. For $\Delta - J = 2$, for instance, we have for the untwisted sector

$$\sum_{l=0}^{J} \mathrm{Tr}[S^q \mathcal{Z}^l (\mathcal{D}_\mu \mathcal{Z}) \mathcal{Z}^{J-l} (\mathcal{D}_\nu \mathcal{Z})] \exp\left(\frac{2\pi \mathrm{i} l}{J} n\right) \quad \leftrightarrow \quad a_n^{\dagger\mu} a_{-n}^{\dagger\nu}|0, p^+\rangle_{q=0}, \tag{2.12}$$

where n is the level of the oscillator $a_n^\dagger$. For the twisted sectors, the correspondence is given by

$$\sum_{l=0}^{J} \mathrm{Tr}[S^q \mathcal{Z}^l \Phi^r \mathcal{Z}^{J-l} \overline{\Phi}^s] \exp\left(\frac{2\pi \mathrm{i} l}{J} n(q)\right) \quad \leftrightarrow \quad \alpha_{n(q)}^{\dagger r} \overline{\alpha}_{-n(q)}^{\dagger s}|0, p^+\rangle_q, \tag{2.13}$$

with $n(q) = n + \frac{q}{k}$.

We shall continue the KPRT analysis [22] in the following, with particular emphasis on the brane engineering of their quiver CFT$_4$. Moreover, the KPRT orbifold model is not unique. We shall thus extend their construction to the entire class of $\mathcal{N} = 2$ quiver CFT$_4$s, thereby covering also the cases with fundamental matter described by D7-branes.

2.1.2. Orientifold. The other important extension of the BMN construction concerns strings moving on a pp-wave orientifold background [24]. There, one considers an $\mathcal{N} = 2 Sp(N)$ gauge theory with a hypermultiplet in the anti-symmetric representation and four fundamental hypermultiplets with open strings in the 't Hooft limit [1]. This is dual to type IIB superstring theory on AdS$_5 \times S^5/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is an orientifold action [23, 24]. As before, one looks for operators that carry large charges with respect to J while their conformal dimensions are such that $\Delta - J$ is small. They are then identified with string states propagating on an orientifold of a maximally supersymmetric plane-wave background [11, 12]. One thereby relates closed strings to single-trace gauge-invariant operators and open strings to gauge-invariant operators with two quarks at the ends. Underlying the construction in [24] is the $O(7)$ orientifold of the pp-wave solution of ten-dimensional type IIB supergravity discussed in [11, 12]. An $O(7)$ plane carries -4 units of D7-brane charge, so the introduction of four D7-branes can cancel this charge, locally producing the gauge group $SO(8)$. The metric of this orientifold background corresponds to (2.5) after the substitution $w_{7,8} \to -w_{7,8}$.

In a field theory with gauge group $Sp(N)$, one has a vector multiplet $\left(0^2, 1, \frac{1}{2}^2\right)$ whose scalars, denoted by the complex fields W_{ab} in the symmetric representation of the gauge group, describe the movement of D3-branes in the (7–8) plane. Matter fields belonging to anti-symmetric hypermultiplets $\left(0^4, \frac{1}{2}^2\right)$, splitting into $\mathcal{N} = 1$ multiplets as $\left(0^2, \frac{1}{2}\right) \oplus \left(0^2, \frac{1}{2}\right)$, are denoted by Z and Z'. They describe the movement of D3-branes in the 3, 4, 5, 6 directions. The fundamental representations of the gauge group associated with these directions are denoted by q_i and $\widetilde{q}_i$, and encode open strings stretching between D3- and D7-branes. The chiral operator that should be identified with the closed-string ground states for $\Delta - J = 0$ is

$$\mathrm{Tr}[(Z\Omega)^J] = (Z_{ab}\Omega^{ba})^J, \tag{2.14}$$

where Ω is an $Sp(N)$ invariant tensor. In this way, the gauge-invariant field operator corresponding to the open-string ground state for $\Delta - J = 1$ reads

$$\mathrm{Tr}[Q_i \Omega (Z\Omega)^J Q_j], \tag{2.15}$$

where the chiral multiplets Q_i, $i = 1, \ldots, 8$, are any of the four fundamentals q and $\widetilde{q}$. Thus, we have the following heuristic rule:

Coordinates	Field insertions
$w^i = x^i, \quad i = 1, \ldots, 4$	$\partial_i Z$
$w^{5,6} = y^{1,2}$	$Z, \overline{Z}'$
$w^{7,8} = y^{3,4}$	$W, \overline{W}$

(2.16)

For non-zero modes we have

$$\sum_{l=0}^{J} \exp\left(\frac{\mathrm{i}2\pi l}{J} n\right) \mathrm{Tr}[W\Omega(Z\Omega)^l Z'\Omega(Z\Omega)^{J-l}] \leftrightarrow \left(a_{-n}^{(7+\mathrm{i}8)\dagger} a_n^{(5+\mathrm{i}6)\dagger} - a_n^{(7+\mathrm{i}8)\dagger} a_{-n}^{(5+\mathrm{i}6)\dagger}\right)|0, p^+\rangle_{\mathrm{lc}}, \tag{2.17}$$

where $a^{(k+\mathrm{i}l)} = a^k + \mathrm{i}a^l$.

2.2. *Outline of construction*

As already mentioned, we wish to study extensions of the BMN correspondence in which type IIB superstrings on a Penrose limit of $\mathrm{AdS}_5 \times S^5$ orbifolded by Γ have four-dimensional $\mathcal{N} = 2$ superconformal field theories as their dual gauge models. We shall proceed in steps by first studying the Penrose limit of type IIB superstrings on $\mathrm{AdS}_5 \times S^5/\Gamma$, where $\Gamma \subset SU(2)$. We then use the AdS/CFT correspondence to explore their various $\mathcal{N} = 2$ CFT_4 duals. Following this, we give brane interpretations of these dual models. We finally consider extensions of the BMN proposal for this class of supersymmetric models.

Our analysis obviously depends on the discrete orbifold group Γ. To indicate how and in order to outline our approach, we shall comment briefly on some of the steps below. More detailed discussions are found in the subsequent sections and in the appendices.

2.2.1. Comments on string-theory side. To derive pp-wave backgrounds in the Penrose limit of $\mathrm{AdS}_5 \times S^5/\Gamma$, we must specify the nature of the orbifold group Γ. This is in general contained in the $SO(6) \sim SU(4)$ R-symmetry of $\mathcal{N} = 4$ SYM_4. Depending on the number of preserved supercharges, in particular, we have the following situations [27]:

$$\begin{array}{ll} \Gamma \subset SU(4), & \Gamma \not\subset SU(3) \subset SU(4), \\ \Gamma \subset SU(3) \subset SU(4), & \Gamma \not\subset SU(2) \subset SU(4), \\ \Gamma \subset SU(2) \subset SU(4). & \end{array} \tag{2.18}$$

For orbifold models with 16 supercharges, one should consider $\Gamma \subset SU(2)$. Using further group-theoretical arguments based on the classification of discrete subgroups of $SU(2)$, it follows that Γ may be any of the discrete $\widetilde{\mathrm{ADE}}$ subgroups[6] of $SU(2)$ [29, 30]. These finite groups are either Abelian or non-Abelian and are classified as follows:

[6] This $\widetilde{\mathrm{ADE}}$ appellation for discrete groups should not be confused with the usual notation for ordinary ADE and affine $\widehat{\mathrm{ADE}}$ Lie algebras.

Γ	Name	Order $\|\Gamma\|$	Group generators
$\widetilde{A}_{k-1} \simeq \mathbb{Z}_k$	Cyclic	k	$\{a \| a^k = \mathrm{Id}\}$
$\widetilde{D}_{2k}$	Dihedral	$4k$	$\{a, b \| b^2 = a^k; ab = ba^{-1}; a^{2k} = \mathrm{Id}\}$
$\widetilde{E}_6$	Tetrahedral	24	$\{a, b \| a^3 = b^3 = (ab)^3\}$
$\widetilde{E}_7$	Octahedral	48	$\{a, b \| a^4 = b^2 = (ab)^2\}$
$\widetilde{E}_8$	Icosahedral	120	$\{a, b \| a^5 = b^3 = (ab)^2\}$

(2.19)

From this classification, one sees that $\widetilde{A}_{k-1} \simeq \mathbb{Z}_k$ is an Abelian group with one generator a, while $\widetilde{D}_{2k}$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ are non-Abelian discrete groups with two non-commuting generators a and b.

There are as many pp-wave geometries with the properties specified by this classification as there are orbifold groups in table (2.19). These geometries are naturally referred to as $\widetilde{\mathrm{ADE}}$ pp-wave geometries.

In the presence of D5–D5 and D3–D5 open-string states, the closed-string geometries have a more general form following from the resolution of the orbifold singularities. Details on the D5–D5 and D3–D5 systems, as well as an implementation of D5–D7 open strings, will be discussed in section 5.

2.2.2. Comments on field-theory side. In order to describe the duality between $\mathcal{N} = 2$ CFT$_4$s and type IIB superstring theory on AdS$_5 \times S^5/\Gamma$, we now identify the various $\mathcal{N} = 2$ CFT$_4$s.

$\mathcal{N} = 2$ CFTs in four dimensions. To get the appropriate $\mathcal{N} = 2$ CFT$_4$s we are interested in, we use results on conformal field theories in four dimensions obtained by taking orbifolds of $\mathcal{N} = 4 U(N)$ SYM$_4$ [27]. As indicated above, we restrict ourselves to the case where Γ is a discrete subgroup of $SU(2)$, itself a subgroup of the $SU(4) R$-symmetry. In this way, $\mathcal{N} = 4$ supersymmetry is broken down to $\mathcal{N} = 2$ supersymmetry, while the original $U(N)$ gauge group gets promoted to $U(|\Gamma|N)$. The $\mathcal{N} = 4$ SYM$_4$ multiplet

$$\left(0^6, \tfrac{1}{2}^4, 1\right) \otimes (|\Gamma|N, \overline{|\Gamma|N}) \tag{2.20}$$

has $6(|\Gamma|N)^2 + 2(|\Gamma|N)^2 = 8(|\Gamma|N)^2$ bosonic degrees of freedom and the same number of fermionic ones. It splits into $\mathcal{N} = 2$ representations as follows:

$$\left(\sum_{i=1}^{|\Gamma|} \left(0^2, \frac{1}{2}^2, 1\right) \otimes (N_i, \overline{N}_i)\right) \oplus \left(\sum_{i,j=1}^{|\Gamma|} \left(0^4, \frac{1}{2}^2\right) \otimes (N_i, \overline{N}_j)\right). \tag{2.21}$$

In this decomposition, the $U(|\Gamma|N)$ gauge group is broken down to $\left[\otimes_{i=1}^{|\Gamma|} U(N_i)\right]$, where

$$|\Gamma|N = \sum_{i=1}^{|\Gamma|} N_i, \tag{2.22}$$

and the one-loop beta function for each gauge coupling g_i is proportional to

$$\beta_i = \frac{1}{6}\left(22N_i - \sum_{j=1}^{|\Gamma|} \left[2\left(a_{ij}^4 + \overline{a}_{ij}^4\right) + a_{ij}^6\right] N_j\right). \tag{2.23}$$

Here a_{ij}^4 and a_{ij}^6 are the numbers of Weyl spinors and scalars, respectively, transforming in the bi-fundamental representations $(N_i, \overline{N}_j)$ [27]. These numbers satisfy

$$3a_{ij}^4 = 3\overline{a}_{ij}^4 = 2a_{ij}^6. \tag{2.24}$$

As we shall see, this relation plays a crucial role in our analysis. For now, we merely mention that (i) the solutions for $\beta_i = 0$ are related to the Dynkin diagrams of affine $\widehat{\text{ADE}}$ Kac–Moody algebras [31–34], and (ii) these solutions may be extended to the class of field theories based on finite and indefinite Lie algebras [35, 36] as well. This property reflects the fact that these classes of field theories can be treated in a unified way.

Geometric engineering of $\mathcal{N} = 2$ QFT$_4$. A powerful method to deal with the classification of these CFT$_4$s is geometric engineering of $\mathcal{N} = 2$ quiver QFT$_4$ developed in [31]; see also [32–34]. It relies on the description and treatment of singular complex manifolds in algebraic geometry. Field-theoretical quantities such as mass and coupling-constant moduli are encoded in equations describing Calabi–Yau manifolds as K3 fibrations. Interesting cases are given by the following singular complex surfaces:

Singularity	Geometry of ALE space	
A_{k-1}	$\zeta_1^2 + \zeta_2^2 = \zeta^k,$	$k \geqslant 2$
D_k	$\zeta_1^2 + \zeta_2^2\zeta = \zeta^{k-1},$	$k \geqslant 3$
E_6	$\zeta_1^2 + \zeta_2^3 + \zeta^4 = 0$	
E_7	$\zeta_1^2 + \zeta_2^3 + \zeta_2\zeta^3 = 0$	
E_8	$\zeta_1^2 + \zeta_2^3 + \zeta^5 = 0$	

(2.25)

Here ζ_1, ζ_2, ζ are complex variables. In general, one is interested in the deformation of the singularities as they are associated with a large class of $\mathcal{N} = 2$ quiver QFT$_4$s. The singular surfaces (2.25), which are associated with the cases where the gauge symmetry of the $\mathcal{N} = 2$ QFT$_4$ is unbroken, have affine extensions classified by affine Lie algebras $\widehat{A}_k$, $\widehat{D}_k$ and $\widehat{E}_s$.

It is recalled that deformations of the singularities can be obtained in two ways, either by Kähler deformations or by deforming the complex structure. In the second case, the complex deformations are carried by complex moduli a_i and the singular surfaces $F(\zeta_1, \zeta_2, \zeta) = 0$ are replaced by non-singular surfaces described by equations of the type

$$F(\zeta_1, \zeta_2, \zeta; \{a_i\}) = 0. \tag{2.26}$$

Following [31, 37], the complex moduli are in general polynomials depending on two extra complex variables, ξ and υ,

$$a_i(\xi, \upsilon) = \sum_{n=1}^{N_i} \sum_{m=1}^{M_i} c_{i_{n,m}} \xi^n \upsilon^m. \tag{2.27}$$

These extra variables allow one to engineer gauge groups and include fundamental matter, where N_i and M_i characterize the gauge groups and their associated matter content.

As an illustration of (2.26), we recall the complex deformation of the ALE surface with A_{k-1} singularity (2.25)

$$z_1 z_2 = \zeta^{k+1} + \sum_{i=1}^{k} a_i \zeta^{k-i}, \tag{2.28}$$

where $z_1 = \zeta_1 + \mathrm{i}\zeta_2$ and $z_2 = \zeta_1 - \mathrm{i}\zeta_2$. One can then relate the various differentials as in the following example:

$$\mathrm{d}z_2 = -\left(\zeta^k + \sum_{i=1}^{k} a_i \zeta^{k-i}\right) \frac{\mathrm{d}z_1}{z_1^2} + \left(k\zeta^{k-1} + \sum_{i=1}^{k} (k-i) a_i \zeta^{k-1-i}\right) \frac{\mathrm{d}\zeta}{z_1} + \sum_{i=1}^{k} \zeta^{k-i}\,\mathrm{d}a_i. \tag{2.29}$$

The terms depending on the differentials $\mathrm{d}a_i$ are important in the metric building of pp-wave orbifolds based on deformations of ADE singularities. In this formulation, the vanishing of the one-loop beta function of these $\mathcal{N} = 2$ quiver QFT$_4$s is translated into a well-known problem of the classification of Lie algebras [31, 35, 36]. Based on these results, we shall show explicitly how one can get the general classes of $\mathcal{N} = 2$ quiver CFT_4s, in particular those involving affine $\widehat{\mathrm{ADE}}$ and finite ADE geometries. Since mainly the first class is well studied in the literature, we take this opportunity to present some explicit results for the finite ADE symmetries.

2.2.3. Brane realizations. As in the case of $\mathcal{N} = 4\, U(N)$ CFT_4s living in the worldvolume of N coincident D3-branes, $\mathcal{N} = 2$ quiver CFT_4s based on Γ orbifolds involve closed strings. The new feature is that the $\mathcal{N} = 2$ models also involve open-string sectors, not present in the $\mathcal{N} = 4$ models. Indeed, brane engineering of $\mathcal{N} = 2$ quiver CFT_4s is based on (partially) wrapped D5- and D7-branes on 2- and 4-cycles, respectively, in addition to the familiar D3-branes. As the initial gauge symmetry is broken by the deformation of the orbifold singularity, open strings are stretched between some of the D-branes. On the field-theory side, this corresponds to $\mathcal{N} = 2$ CFT_4s with *bi-fundamental* matter, which has no analogue in $\mathcal{N} = 4$ CFT_4.

With the above tools at hand, we can write down the correspondence rule between string states on pp waves on $\mathrm{AdS}_5 \times S^5/\Gamma$, where Γ is a discrete subgroup of $SU(2)$, and gauge-invariant field operators in the dual $\mathcal{N} = 2$ CFT_4s. As we will show, the resulting correspondence rule for $\mathcal{N} = 2$ quiver gauge theories is richer than that indicated in (2.10)–(2.13).

3. Type IIB superstrings on pp-wave orbifolds

In this section, we consider the problem of metric building of pp waves on $\mathrm{AdS}_5 \times S^5/\Gamma$ orbifolds with $\Gamma \subset SU(2)$. As Γ may be any of the discrete groups $\widetilde{A}_{k-1}$, $\widetilde{D}_{2k}$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$, we will study each case separately and treat both the singular and deformed geometries. For an explicit treatment, we will first consider the Abelian group orbifolds where $\Gamma = \widetilde{A}_{k-1}$. The results for the non-Abelian cases and the results for the affine geometries are deferred to appendix A.

3.1. Metric of Abelian orbifolds

Following the previous section, we shall use the metric

$$\mathrm{d}s^2 = -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |z_1|^2 + |z_2|^2)(\mathrm{d}x^+)^2 + |\mathrm{d}z_1|^2 + |\mathrm{d}z_2|^2 \tag{3.1}$$

to describe the Penrose limit of $\mathrm{AdS}_5 \times S^5$. The string coupling $g_s \sim g^2_{\mathrm{YM}}$ of the type IIB superstring theory is kept fixed in this limit, while the curvature, R, of the ten-dimensional background is related to the gauge group $U(N)$ of the dual four-dimensional SYM_4 through $R^4 \sim \left(g^2_{\mathrm{YM}}N\right)l_s^4$. For non-zero μ, the metric (3.1) is invariant under the action of $SO(4) \otimes SO(4)$ on the eight coordinates $(\mathbf{x}, \mathbf{y})$ of $\mathbb{R}^4 \times \mathbb{R}^4$. The μ-dependent term proportional to $(\mathrm{d}x^+)^2$ manifestly breaks the $SO(1,1)$ subgroup of the symmetry group $SO(1,9)$ of the ten-dimensional space with $\mu = 0$. For small values of μ, the metric (3.1) thus corresponds to a deformation of the flat spacetime background $\mathbb{R}^{1,9}$. After replacing $\mathbb{R}^4 \times \mathbb{R}^4$ by $\mathbb{R}^4 \times \mathbb{C}^2$, the group $SO(4) \otimes U(2)$ naturally plays the role of $SO(4) \otimes SO(4)$.

The space $\mathrm{AdS}_5 \times S^5$ is known to be an exactly solvable, maximally supersymmetric geometry preserving the 32 spacetime supercharges [11, 12]. This preserving property survives in the Penrose limit. As discussed above, 16 supercharges may be reached by considering type IIB superstrings on orbifolds like $\mathrm{AdS}_5 \times S^5/\Gamma$ in certain Penrose limits. Here we focus on Penrose limits of the orbifolds $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_k$, while the other geometries are discussed in appendix A. To work out the explicit form of the metric, let us list a couple of observations. (i) Under orbifolding by $\mathbb{Z}_k$, the original supersymmetry of $\mathcal{N} = 4\,U(N)$ SYM_4 is reduced by half, whereas the gauge group is promoted to $U(kN)$. The first property implies that the massless $\mathcal{N} = 4$ on-shell gauge multiplet $\left(0^6, \frac{1}{2}^4, 1\right)$ splits into $\mathcal{N} = 2$ representations according to

$$\left(0^6, \tfrac{1}{2}^4, 1\right) \to \left(0^2, \tfrac{1}{2}^2, 1\right) \oplus \left(0^4, \tfrac{1}{2}^2\right), \tag{3.2}$$

where $\left(0^2, \frac{1}{2}^2, 1\right)$ is an $\mathcal{N} = 2$ vector multiplet and $\left(0^4, \frac{1}{2}^2\right)$ is a hypermultiplet. Under this decomposition, the initial $SO(6) \sim SU(4)\, R$-symmetry of the $\mathcal{N} = 4$ theory is broken down to $SO(2) \otimes SO(4)$ which in terms of complex parameters corresponds to $U(1) \otimes SU(2) \otimes SU(2)$. This group contains the usual $U(2) \sim U(1) \otimes SU(2)\, R$-symmetry of $\mathcal{N} = 2$ gauge theories in four dimensions. (ii) Under orbifolding, the transverse light-cone space $\mathbb{R}^4 \times \mathbb{R}^4$ is now replaced by the orbifold $\mathbb{R}^4 \times (\mathbb{R}^4/\mathbb{Z}_k)$.

The action of the orbifold group given in (2.8) has a fixed point at $z_1 = z_2 = 0$. This means that quantum fields and string states at this point have k sectors. To determine the metric of this pp-wave background, one should understand the geometry in the vicinity of the orbifold (fixed) point. In this regard, we recall that the real four-dimensional orbifold $\mathbb{R}^4/\mathbb{Z}_k$ near the singular point $y^1 = y^2 = y^3 = y^4 = 0$ looks like the ALE space with an A_{k-1} singularity which can be embedded in a three-dimensional complex plane. This is a crucial point in our construction, and it is important here to specify the nature of this singularity which can be either elliptic, ordinary or indefinite [35, 36]. In the elliptic case, one has only closed-string states while for the other two geometries one needs open strings as well. On the field-theory side, this corresponds to adding fundamental matter in order to ensure the vanishing of the beta function. Put differently, one needs complex moduli a_i with non-zero differentials $\mathrm{d}a_i$ as in (2.26). To show how the machinery works, we start by treating orbifolds with an ordinary A_{k-1} singularity and initially disregard the open strings. The rationale for starting with this example is that it is sufficiently simple to handle while it nevertheless illustrates the implementation of partially wrapped D-branes. Once we get the explicit form of the pp-wave metric with A_{k-1} geometry, open strings can be introduced in conjunction with the complex moduli (2.27).

It is recalled that an ALE space with an A_{k-1} singularity is a complex surface embedded in $\mathbb{C}^3$ and may be defined by the algebraic equation

$$zt = \zeta^k. \tag{3.3}$$

The metric of this complex space is induced from $\mathrm{d}s^2 = |\mathrm{d}z|^2 + |\mathrm{d}t|^2 + |\mathrm{d}\zeta|^2$ by substituting (3.3). In terms of the coordinates $z_1 = z$ and $z_2 = \frac{\zeta^k}{z}$, the differentials are related as follows:

$$\mathrm{d}z_2 = k\frac{\zeta^{k-1}}{z}\,\mathrm{d}\zeta - \frac{\zeta^k}{z^2}\,\mathrm{d}z. \tag{3.4}$$

It is noted that $\mathbb{Z}_k$ acts on these variables as $z \to \alpha_z z$, $t \to \alpha_t t$ and $\zeta \to \alpha_\zeta \zeta$, where α_z, α_t and α_ζ are kth roots of unity. Invariance of (3.3) requires $\alpha_t = \overline{\alpha}_z$ while α_ζ can be any of the kth roots of unity. As in (2.8), we shall work with

$$z_1 \to \omega z_1, \qquad z_2 \to \overline{\omega} z_2, \qquad \zeta \to \overline{\omega}\zeta, \qquad w = \exp\left(\frac{2\pi \mathrm{i}}{k}\right), \tag{3.5}$$

where we have set $\alpha_\zeta = \overline{\omega}$. This is a convenient choice to be used below in the classification of the possible deformations.

Returning to the pp-wave orbifold (3.1), we thus find that its metric near the A_{k-1} singularity reads

$$\begin{aligned}\mathrm{d}s^2|_{A_{k-1}} = {}& -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2\left(\mathbf{x}^2 + |z|^2 + \left|\frac{\zeta^k}{z}\right|^2\right)(\mathrm{d}x^+)^2 + \left(1 + \left|\frac{\zeta^k}{z^2}\right|^2\right)|\mathrm{d}z|^2 \\ & + k^2\frac{|\zeta|^{2(k-1)}}{|z|^2}|\mathrm{d}\zeta|^2 - k\frac{|\zeta|^{2(k-1)}}{|z|^4}[\zeta\overline{z}\,\mathrm{d}z\,\mathrm{d}\overline{\zeta} + z\overline{\zeta}\,\mathrm{d}\zeta\,\mathrm{d}\overline{z}].\end{aligned} \tag{3.6}$$

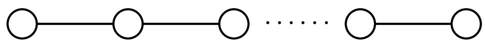
The singularity of the orbifold is resolved by replacing it by a complex two-dimensional manifold representing the deformed geometry near the origin of the A_{k-1} ALE space [38]. Since there are two mirror ways of deforming the singularity, namely Kähler and complex deformations, we will study both of them in the following and work out the corresponding non-singular pp-wave metrics. For the time being, we note that type IIB superstring theory in the Penrose limit of $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_k$ involves kN D5-branes (partially) wrapped on a system of shrunken 2-cycles. The worldvolume variables of these wrapped D5-branes are given by the local variables $\{x^+, x^-, x^1, x^2\}$ as indicated in the following table:

Coordinate	x^+	x^-	x^1	x^2	x^3	x^4	$z_1 = z$	$z_2 = \zeta^k/z$
Wrapped D5-branes	$\surd$	$\surd$	$\surd$	$\surd$	–	–	–	–
$\mathbb{Z}_k$ charge	0	0	0	0	0	0	1	-1

(3.7)

3.2. PP waves on orbifolds

3.2.1. Orbifolds with Kähler deformations. The Kähler resolution of the singular A_{k-1} geometry is obtained by blowing up the singular point using $k-1$ intersecting 2-spheres, $\mathbb{CP}^1 \sim S^2$, arranged as shown here

A_{k-1} :  (3.8)

where the nodes represent $\mathbb{CP}^1$ curves, while their intersections are represented by the links. To describe the blow-up of the A_{k-1} singularity, one may introduce k complex two-dimensional planes $\mathcal{U}_i$ parameterized by (u_i, v_i) [38]. In this way, the coordinates of the original singular manifold are realized as

$$z_1 = u_i^i v_i^{i-1}, \qquad z_2 = u_i^{k-i} v_i^{k+1-i}, \qquad \zeta = u_i v_i. \tag{3.9}$$

The transition functions describing how the patches are glued together are given by

$$v_i u_{i+1} = 1, \qquad u_i v_i = u_{i+1} v_{i+1}. \tag{3.10}$$

Combined, the variables $\{u_i, v_i\}$ describe a system of $k-1$ intersecting rational curves, $\mathcal{C}_i$, given by

$$\begin{aligned} &\mathcal{C}_i = \{v_i u_{i+1} = 1, u_i = v_{i+1} = 0\}, \qquad i = 1, \ldots, k-1, \\ &\mathcal{C}_i \cap \mathcal{C}_j = \varnothing, \qquad \text{unless} \quad j = i \pm 1, \\ &\mathcal{C}_i \cap \mathcal{C}_{i+1} = \{u_{i+1} = v_{i+1} = 0\}. \end{aligned} \tag{3.11}$$

The manifold characterized by these intersecting curves is smooth and mapped isomorphically into the singular A_{k-1} surface, except at the inverse image of the singular point $(z_1, z_2, \zeta) = (0, 0, 0)$.

From relations (3.10) and the transformations of z_1, z_2 and ζ in (3.5), one sees that the orbifold group $\mathbb{Z}_k$ acts on the variables u_i and v_i as

$$\mathbb{Z}_k: \quad u_i \mapsto \omega^i u_i, \qquad v_i \mapsto \overline{\omega}^{i+1} v_i. \tag{3.12}$$

Based on (3.9), one can write down the pp-wave metric of the Penrose limit of type IIB superstrings on the blown-up singularity of $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_k$. On the jth patch $\mathcal{U}_j$, the differentials $\mathrm{d}z_1, \mathrm{d}z_2$ and $\mathrm{d}\zeta$ may be expressed in terms of $\mathrm{d}u_j$ and $\mathrm{d}v_j$ as

$$\begin{aligned} \mathrm{d}z_1 &= j u_j^{j-1} v_j^{j-1}\, \mathrm{d}u_j + (j-1) u_j^j v_j^{j-2}\, \mathrm{d}v_j, \\ \mathrm{d}z_2 &= (k-j) u_j^{k-j-1} v_j^{k+1-j}\, \mathrm{d}u_j + (k+1-j) u_j^{k-j} v_j^{k-j}\, \mathrm{d}v_j, \\ \mathrm{d}\zeta &= \mathrm{d}u_j v_j + u_j\, \mathrm{d}v_j. \end{aligned} \tag{3.13}$$

The metric of the jth patch $\mathcal{U}_j$ now follows directly from (3.1) by substituting relations (3.9) and expressions (3.13) for $\mathrm{d}z_1$ and $\mathrm{d}z_2$. We find that

$$\begin{aligned} \mathrm{d}s^2|_{A_{k-1},\mathcal{U}_j} = {} & -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |u_j|^{2j}|v_j|^{2j-2} + |u_j|^{2k-2j}|v_j|^{2k+2-2j})(\mathrm{d}x^+)^2 \\ & + \{j^2|u_j|^{2j-2}|v_j|^{2j-2} + (k-j)^2|u_j|^{2k-2j-2}|v_j|^{2k-2j+2}\}|\mathrm{d}u_j|^2 \\ & + \{(j-1)^2|u_j|^{2j}|v_j|^{2j-4} + (k-j+1)^2|u_j|^{2k-2j}|v_j|^{2k-2j}\}|\mathrm{d}v_j|^2 \\ & + \{j(j-1)|u_j|^{2j-2}|v_j|^{2j-4} + (k-j)(k-j+1)|u_j|^{2k-2j-2}|v_j|^{2k-2j}\} \\ & \times (\overline{u}_j v_j\,\mathrm{d}u_j\, d\overline{v}_j + u_j\overline{v}_j\,\mathrm{d}\overline{u}_j\,\mathrm{d}v_j). \end{aligned} \tag{3.14}$$

In the special case of a $\mathbb{Z}_2$ orbifold, we have two coordinate patches, $\mathcal{U}_1$ and $\mathcal{U}_2$, to describe the blown-up geometry. The metric of the resolved pp wave based on the coordinate patch $\mathcal{U}_1$ reads

$$\begin{aligned} \mathrm{d}s^2|_{A_1,\mathcal{U}_1} = {} & -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |u_1|^2 + |u_1|^2|v_1|^4)(\mathrm{d}x^+)^2 + (1+|v_1|^4)|\mathrm{d}u_1|^2 \\ & + 4|u_1|^2|v_1|^2|\mathrm{d}v_1|^2 + 2|v_1|^2(\overline{u}_1 v_1\,\mathrm{d}u_1\,\mathrm{d}\overline{v}_1 + u_1\overline{v}_1\,\mathrm{d}\overline{u}_1\,\mathrm{d}v_1), \end{aligned} \tag{3.15}$$

while the metric based on the coordinate patch $\mathcal{U}_2$ reads

$$\begin{aligned} \mathrm{d}s^2|_{A_1,\mathcal{U}_2} = {} & -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |u_2|^4|v_2|^2 + |v_2|^2)(\mathrm{d}x^+)^2 + 4|u_2|^2|v_2|^2|\mathrm{d}u_2|^2 \\ & + (1+|u_2|^4)|\mathrm{d}v_2|^2 + 2|u_2|^2(\overline{u}_2 v_2\,\mathrm{d}u_2\,\mathrm{d}\overline{v}_2 + u_2\overline{v}_2\,\mathrm{d}\overline{u}_2\,\mathrm{d}v_2). \end{aligned} \tag{3.16}$$

It is recalled that the transition functions between the two realizations (3.15) and (3.16) of the metric $\mathrm{d}s^2$ of the pp wave are given by (3.10).

On the blown-up manifold, the original kN wrapped D5-branes are now partitioned into $k-1$ subsets of N_i D5-branes at the origins of the rational curves, $\mathcal{U}_i$, and the original gauge group $U(kN)$ breaks down according to

$$U(kN) \to \left[\otimes_{i=1}^{k-1} U(N_i)\right], \tag{3.17}$$

where

$$kN = \sum_{i=1}^{k-1} N_i. \tag{3.18}$$

Just as z_1 and z_2, the new coordinates $\{u_i, v_i\}$ have superpartners. They will be discussed in section 4 when we study supersymmetric blow-ups and in the derivation of an extended BMN correspondence for ADE orbifolds.

3.2.2. Orbifolds with complex deformations. In mirror geometry, the deformation of the A_{k-1} singularity (3.3) is obtained by introducing $k+1$ complex variables τ_i satisfying the following relation [31, 37]:

$$\tau_i\tau_{i+2} = \tau_{i+1}^2. \tag{3.19}$$

These constraint equations can also be put into a more convenient form as

$$\prod_{i=1}^{k+1} \tau_i^{\ell_i^{(a)}} = 1, \qquad a = 1, \ldots, k-1, \tag{3.20}$$

where $\ell^{(a)}$ are integer vectors given by

$$\begin{aligned} \ell^{(1)} &= (1, -2, 1, 0, 0, 0, \ldots, 0), \\ \ell^{(2)} &= (0, 1, -2, 1, 0, 0, \ldots, 0), \\ &\vdots \\ \ell^{(k-1)} &= (0, 0, 0, 0, \ldots, 1, -2, 1). \end{aligned} \tag{3.21}$$

These vectors are recognized as a simple extension of minus the Cartan matrix K_{ab} of the A_{k-1} Lie algebra. As far as explicit realizations of τ_i are concerned, (3.19) may be solved naturally in terms of monomials of two independent complex variables η and ξ as follows:

$$\tau_i = \eta^{k+1-i}\xi^{i-1} = \eta^k\zeta^{i-1}, \qquad i = 1, \ldots, k+1, \qquad \zeta = \frac{\xi}{\eta}. \tag{3.22}$$

In toric geometry [31], the monomials ζ^i are associated with the compact curves $\mathcal{C}_i$ of (3.11), and the quiver graph for the mirror geometry is the Dynkin diagram of the A_{k-1} Lie algebra as shown in (3.8). By varying the homogeneous factor, one may in general fix one of the variables η or ξ. In the particular case where $\eta = 1$, which corresponds to $\tau_1 = 1$, it follows from (3.22) that $\tau_i = \xi^{i-1}$. It is also noted that

$$\tau_i \to \omega^{i-1}\tau_i, \qquad i = 1, \ldots, k+1, \tag{3.23}$$

provides a natural representation of the action of $\mathbb{Z}_k$ on the variables τ_i, cf (3.19).

To get the pp-wave metric associated with the complex deformation of A_{k-1} singularity, one may consider the situation where the singularity is partially solved (see (2.28)) by taking

$$z_1 z_2 = \zeta^k + \sum_{i=1}^{k-j} a_i \zeta^{k-i}, \qquad 1 \leqslant j \leqslant k, \tag{3.24}$$

where $z_1 = \eta^k$ and $z_2 = \tau_{j+1}$. In this case, the N_j D5-branes wrapping the spheres represented by jth node $\mathcal{O}_j$ of the complex deformation of the A_{k-1} singularity are parameterized as indicated in the following table:

Coordinate	x^+	x^-	x^1	x^2	x^3	x^4	z_1	τ_j
Wrapped D5-branes	$\surd$	$\surd$	$\surd$	$\surd$	$-$	$-$	$-$	$-$
$\mathbb{Z}_k$ charge	0	0	0	0	0	0	1	$j-1$

(3.25)

Using these mirror complex variables, we have $|z_1|^2 = |\eta|^{2k}$, $|\tau_j|^2 = |\eta|^{2k}|\zeta|^{2j}$ and

$$\mathrm{d}z_1 = k\eta^{k-1}\,\mathrm{d}\eta, \qquad \mathrm{d}z_2 = \zeta^j\,\mathrm{d}z_1 + jz_1\zeta^{j-1}\,\mathrm{d}\zeta. \tag{3.26}$$

The metric of the pp wave on the deformed geometry that follows from (3.1) by substituting the expressions of τ_j in terms of z_1 and ζ reads

$$\begin{aligned}\mathrm{d}s^2|_{A_{k-1},\mathcal{O}_j} = {}& -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |z_1|^2(1+|\zeta|^{2j}))(\mathrm{d}x^+)^2 + (1+|\zeta|^{2j})|\mathrm{d}z_1|^2 \\ &+ j^2|z_1|^2|\zeta|^{2(j-1)}|\mathrm{d}\zeta|^2 + j|\zeta|^{2(j-1)}(\zeta\bar{z}_1\,\mathrm{d}z_1\,\mathrm{d}\bar{\zeta} + z_1\bar{\zeta}\,\mathrm{d}\zeta\,\mathrm{d}\bar{z}_1).\end{aligned} \tag{3.27}$$

Particular cases are easily described by substituting given values for j.

The above analysis for the A_{k-1} singularity can be extended to the other possible finite and affine singularities of the ALE space. These extensions are addressed in appendix A and are based on a specification of the Kähler and complex properties of the sets $\{\mathcal{O}_j\}$ used to resolve the singularity.

Open-string sector. A way to see how open strings can be implemented is to consider the adjunction of fundamental matter in the dual field theory. To that purpose recall that in the dual four-dimensional $\mathcal{N} = 2$ field theory of the above type IIB superstring on the $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_k$ orbifold, the a_i moduli parameterizing the complex deformation of the singularity are associated with the vacuum expectation values of the scalars of the $\mathcal{N} = 2\left[\otimes_{i=1}^{k-1} U(N_i)\right]$ hypermultiplets $\left(0^4, \frac{1}{2}^2, 1\right)$ in the bi-fundamental representation of $U(N_i) \otimes U(N_{i+1})$. To see how this follows from the original $\mathcal{N} = 4U(kN)$ SYM_4, it is enough to decompose the gauge multiplet

$$\left(0^6, \tfrac{1}{2}^4, 1\right) \otimes (kN, \overline{kN}), \tag{3.28}$$

with $6(kN)^2 + 2(kN)^2 = 8(kN)^2$ bosonic degrees of freedom and as many fermionic ones, on two representations of the orbifold group along the lines of [27]. Under deformation of the A_{k-1} singularity, the orbifold gauge group $U(kN)$ is generally broken down to $\left[\otimes_{i=1}^{k-1} U(N_i)\right]$ with $kN = \sum_{i=1}^{k-1} N_i$ and $\mathcal{N} = 4$ supersymmetry is broken down to $\mathcal{N} = 2$. In this case, the previous $\mathcal{N} = 4$ vector multiplet decomposes as $\mathcal{N} = 2$ gauge multiplets $\left(0^2, \frac{1}{2}^4, 1\right)$ plus hypermultiplets $\left(0^4, \frac{1}{2}^2\right)$. In the $\mathcal{N} = 2$ quiver gauge theory where the A_{k-1} singularity is completely resolved, the $U(kN)$ gauge-group multiplet (3.28) decomposes into $k-1$ $\mathcal{N} = 2$ on-shell gauge multiplets in the adjoint representations of $U(N_i)$

$$\left(0^2, \tfrac{1}{2}^2, 1\right) \otimes (N_i, \overline{N}_i), \qquad i = 1, \ldots, k-1, \tag{3.29}$$

in addition to $k-2$ hypermultiplets $\left(0^4, \frac{1}{2}^2\right)$ in the bi-fundamental representations of the $U(N_i) \otimes U(N_{i+1})$ subgroups of the gauge group $\left[\otimes_{i=1}^{k-1} U(N_i)\right]$ as

$$\left(0^4, \tfrac{1}{2}^2\right) \otimes (N_i, \overline{N}_{i+1}), \qquad i = 1, \ldots, k-2. \tag{3.30}$$

Under deformation of the orbifold point, the initial group symmetry $SO(6) \otimes U(kN)$ breaks down to $SO(2) \otimes SU(2) \otimes \left[\otimes_{i=1}^{k-1} U(N_i)\right]$. In this way, the original six scalar fields $\phi_1, \ldots, \phi_6$ of (3.28), transforming in the $(6, N \otimes \overline{N})$ representation of $SO(6) \otimes U(kN)$, split as follows:

$$(6, kN \otimes k\overline{N}) \longrightarrow \left[\oplus_{i=1}^{k-1} (2, 1, N_i \otimes \overline{N}_i)\right] \oplus \left[\oplus_{i=1}^{k-2} (1, 2, N_i \otimes \overline{N}_{i+1})\right]. \tag{3.31}$$

For the spin-1 gauge field A_μ at the orbifold point, the decomposition that follows after the complex deformation of A_{k-1} singularity reads

$$(1, kN \otimes k\overline{N}) \longrightarrow \left[\oplus_{i=1}^{k-1} (1, 1, N_i \otimes \overline{N}_i)\right]. \tag{3.32}$$

Similarly for the gauginos where we have

$$(4, kN \otimes \overline{kN}) \longrightarrow \left[\oplus_{i=1}^{k-1} (1, 2, N_i \otimes \overline{N}_i)\right] \oplus \left[\oplus_{i=1}^{k-2} (\pm 1, 1, N_i \otimes \overline{N}_{i+1})\right]. \tag{3.33}$$

Note that the first contributions to (3.31) and (3.33) combined with (3.32) make up the $\mathcal{N} = 2$ gauge multiplets in the adjoint representation of $U(N_i)$, while the second contributions to (3.31) and (3.33) give bi-fundamental matter. Note also that under this orbifolding there is no fundamental matter that follows from the reduction of $\mathcal{N} = 4$ SYM$_4$. Finally, while $\mathcal{N} = 4$ SYM$_4$ is scale invariant, its reduction to $\mathcal{N} = 2$ SYM$_4$ is not necessarily a conformal theory since the beta function (2.23) may not vanish. They do, though, for orbifolds associated with $\mathcal{N} = 2$ quiver gauge theories based on affine $\widehat{\text{ADE}}$ singularities, while the models based on finite singularities are not conformal. This is illustrated by the example of ordinary A_{k-1} in which case β_1 and β_{k-1} are non-zero. To recover conformal invariance, one needs the introduction of fundamental matter by allowing the complex moduli to have extra dependencies as in (2.27). On the string-theory side, this corresponds to allowing open-string sectors stretching between wrapped D5-branes and wrapped D7-branes. In section 5, we describe such systems. In section 6, we give the correspondence between type IIB superstring states on pp waves and gauge-invariant operators in the above $\mathcal{N} = 2$ CFT$_4$s, including those involving fundamental matter.

4. $\mathcal{N}$ = 2 superconformal theories in four dimensions

Due to their special features, supersymmetric conformal field theories in various dimensions have been a subject of great interest over the last couple of decades. A particular one of these scale-invariant theories has recently been studied intensively. It is the four-dimensional $\mathcal{N} = 4$ superconformal field theory with both gauge and matter fields in the adjoint representation of the gauge group [1]. A simple counting of the degrees of freedom of $\mathcal{N} = 4\, SU(N)$ QFT$_4$ shows that the following beta function vanishes identically:

$$\beta = \tfrac{11}{3}C(G) - \tfrac{2}{3}T(R) - \tfrac{1}{6}T(S). \tag{4.1}$$

In this equation, $C(G)$ is the Casimir of the gauge group, $T(R)$ is the number of fundamental fermions, while $T(S)$ is the number of adjoint scalars. For $SU(N)$ gauge symmetry, we have $C(G) = N$, $T(R) = 4N$ and $T(S) = 6N$, in which case the beta function indeed vanishes. The $\mathcal{N} = 4SU(N)$ gauge theory is then a critical theory which can be embedded in string theory compactifications preserving 32 supercharges. It also plays a central role in the study of extensions of the BMN proposal.

Besides this standard example, there are several other four-dimensional conformal field theories, though with a lower number of conserved supercharges. These models are essentially obtained from four-dimensional $\mathcal{N} = 4SU(N)$ gauge theory by appropriate deformations preserving a fraction of the 32 original supercharges. Examples are the four-dimensional $\mathcal{N} = 2$ models being obtained as resolutions of orbifolds based on discrete subgroups of $SU(2)$. In these examples, the beta functions β_i (2.23) may be written as

$$\beta_i = \frac{11}{3}\left(N_i - \frac{1}{4}\sum_j a^4_{ij} N_j\right), \tag{4.2}$$

where we have used relations (2.24). The vanishing conditions for these beta functions can be brought to a form familiar from Lie algebra theory, namely

$$\beta_i = \frac{11}{6}\sum_j K_{ij} N_j = 0, \tag{4.3}$$

where the matrix K_{ij} is given by

$$K_{ij} = 2\delta_{ij} - \tfrac{1}{2}a^4_{ij}. \tag{4.4}$$

This matrix is not necessarily linked directly to a standard Cartan matrix, though affine Kac–Moody algebras do provide natural candidates, as we then have simple identities of the form

$$a^4_{ij} = 2\delta_{i,j-1} + 2\delta_{i,j+1}. \tag{4.5}$$

Situations based on finite or indefinite Lie algebras have been considered in [35, 36]. They require adding fundamental matter to the bi-fundamental matter, already present in the affine case, in order to satisfy the vanishing condition of the beta function. The engineering of $\mathcal{N} = 2$ CFT$_4$s therefore depends on the matter content of the gauge theory, where the possible matter contents are linked to the classification theorem of Kac–Moody algebras [39–42]. This theorem classifies the algebras according to three sectors, in particular, namely finite ($q = 1$), affine ($q = 0$) and indefinite ($q = -1$) Lie algebras

$$\sum_j K^{(q)}_{ij} n_j = q m_i, \qquad q = 0, \pm 1, \tag{4.6}$$

where $K^{(q)}_{ij}$ denotes the generalized Cartan matrix, while n_j and m_j are positive integers. One should therefore expect that there are three classes of $\mathcal{N} = 2$ CFT$_4$s [35, 36]. Here we will discuss the two classes associated with ordinary or affine Lie algebras. We leave the third class based on indefinite Lie algebras to future investigations.

Following [31, 32], one may distinguish between two classes of $\mathcal{N} = 2$ superconformal $[\otimes_i SU(N_i)]$ quiver gauge theories in four dimensions according to whether or not the model contains fundamental matter. That is, we have the following two classes of models: (i) four-dimensional $\mathcal{N} = 2$ SCFT with gauge group G_{g} of the form

$$G_{\mathrm{g}} = [\otimes_i SU(N_i)], \tag{4.7}$$

with a special set of bi-fundamental matter $(N_i, \overline{N}_j)$; (ii) four-dimensional $\mathcal{N} = 2$ SCFT with symmetry group G composed of a gauge group G_{g} as in the first class but here with an extra G_{f} flavour symmetry, i.e.,

$$G = G_{\mathrm{g}} \otimes G_{\mathrm{f}}. \tag{4.8}$$

In this case, there is a chain of bi-fundamental matter $(N_i, \overline{N}_j)$ in addition to the fundamental matter (N_l) transforming under the flavour group. The numbers n_{f} and n_{bif} of fundamental and bi-fundamental matter, respectively, cannot be arbitrary because of the condition that the beta function must vanish. Furthermore, n_{f} and n_{bif} depend on the type of the underlying orbifold.

The four-dimensional $\mathcal{N} = 2$ quiver gauge models we will consider in the present study admit an elegant description in terms of the Cartan matrices K and $\widehat{K}$ of ordinary ADE and affine $\widehat{\mathrm{ADE}}$ Lie algebras, respectively, where K refers to $K^{(+)}$ while $\widehat{K}$ corresponds to $K^{(0)}$ in the notation of (4.6). If we suppose that the symmetry group G of the quiver gauge theory is given by (4.8) with gauge group (4.7) and flavour symmetry

$$G_{\mathrm{f}} = [\otimes_l SU(M_l)], \tag{4.9}$$

then the conformal invariance conditions $\beta_i = 0$ can be translated into an algebraic constraint equation on the matter content of the model. More precisely, we have

$$\beta_i = 0 \quad \Leftrightarrow \quad \sum_j \widehat{K}_{ij} N_j = 0 \tag{4.10}$$

in the case of bi-fundamental matter only and

$$\beta_i = 0 \quad \Leftrightarrow \quad \sum_j K_{ij} N_j = M_i \tag{4.11}$$

in the case where one has both bi-fundamental and fundamental matter. These algebraic relations were first noted in the context of geometric engineering of $\mathcal{N} = 2$ QFT$_4$ embedded in type II superstring theory [31, 32]. Here we want to exploit this important observation to discuss the various types of $\mathcal{N} = 2$ CFT$_4$s. Then we use these results to study extensions of the BMN correspondence between gauge-invariant operators of $\mathcal{N} = 2$ CFT$_4$ and type IIB superstring theory on pp waves with ADE geometry. First, though, we shall discuss the solutions to the vanishing beta function conditions with initial focus on (4.10) followed by an analysis of (4.11).

Quiver CFT$_4$s based on indefinite Lie algebras have been studied in [35, 36].

4.1. $\mathcal{N} = 2$ affine CFT$_4$s

We will discuss individually the various $\mathcal{N} = 2$ quiver CFT$_4$s associated with the affine $\widehat{\mathrm{ADE}}$ Kac–Moody algebras. The $\mathcal{N} = 2$ affine $\widehat{A}_k$ CFT$_4$s are analysed in the following, whereas results for $\widehat{D}_k$ and $\widehat{E}_s$ are deferred to appendix A.

In our context, the simplest example of an $\mathcal{N} = 2$ superconformal field theory without fundamental matter is based on the affine Lie algebra $\widehat{A}_k$. This model has no flavour symmetry while the gauge group is $G_{\mathrm{g}} = \left[\otimes_{i=0}^{k} U(N_i)\right]$ subject to $\sum_{i=0}^{k} N_i = (k+1)N$. One may also evaluate N_is explicitly as conformal invariance imposes the condition

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -1 & 2 & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_{k-2} \\ N_{k-1} \\ N_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.12}$$

corresponding to the following system of linear equations:

$$2N_1 - N_2 - N_k = 0, \tag{4.13}$$

$$-N_{j-1} + 2N_j - N_{j+1} = 0, \qquad 2 \leqslant j \leqslant k-1, \tag{4.14}$$

$$-N_1 - N_{k-1} + 2N_k = 0. \tag{4.15}$$

These equations can be solved straightforwardly by

$$N_i = N, \tag{4.16}$$

in which case the gauge group of the $\mathcal{N} = 2$ SCFT$_4$ simply reads

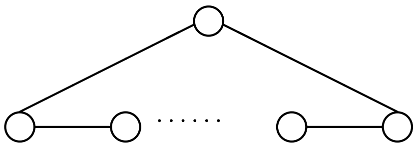
$$G = [U(N)]^{k+1}. \tag{4.17}$$

The set of bi-fundamental matter

$$\{\mathcal{H}_{i,\overline{j}} = (N_i, \overline{N}_j) \mid j - i \equiv 1 \bmod k+1\} \tag{4.18}$$

forms a one-dimensional quiver with $k+1$ edges where an edge like $\mathcal{H}_{i,\overline{i+1}}$ is associated with a hypermultipet in the bi-fundamental of $U(N_i) \otimes U(N_{i+1}) = U(N) \otimes U(N)$. This system is seen to be described by the affine $\widehat{A}_k$ Dynkin diagram

$\widehat{A}_k$:



(4.19)

The complete set of hypermultiplets is thus given by the following chain:

$$\mathcal{H}_{0,\overline{1}}, \mathcal{H}_{1,\overline{2}}, \ldots, \mathcal{H}_{k-1,\overline{k}}, \mathcal{H}_{k,\overline{0}}. \tag{4.20}$$

Each $\mathcal{N} = 2$ hypermultiplet $\mathcal{H}_{i,\overline{i+1}}$ carries $4+4$ on-shell degrees of freedom:

$$\begin{aligned} &\mathcal{H}_{i,\overline{i+1}} = \left(0^4, \tfrac{1}{2}^2\right), \\ &0^4 \sim \phi^{\alpha}_{i,\overline{i+1}}, \qquad \alpha = 1, 2, \\ &\tfrac{1}{2}^2 \sim \psi_{i,\overline{i+1}} \oplus \overline{\chi}_{i,\overline{i+1}}. \end{aligned} \tag{4.21}$$

It may be decomposed into two $\mathcal{N} = 1$ chiral multiplets, here denoted as $Q_{i,\overline{i+1}} \sim (q_{i,\overline{i+1}}, \psi_{i,\overline{i+1}})$ and $\widetilde{Q}_{i,\overline{j}} \sim (\widetilde{q}_{i,\overline{i+1}}, \widetilde{\psi}_{i,\overline{i+1}})$. Their charges under a $U(1)$ subgroup of the $SU(2) \subset SO(4)R$-symmetry are $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. In terms of these multiplets, the matter content of the affine $\widehat{A}_k$ quiver CFT$_4$ reads

$$\begin{pmatrix} Q_{0,\overline{1}} \\ \widetilde{Q}_{0,\overline{1}} \end{pmatrix}, \ldots, \begin{pmatrix} Q_{k,\overline{0}} \\ \widetilde{Q}_{k,\overline{0}} \end{pmatrix}. \tag{4.22}$$

The scalar fields $q_{i,\overline{j}}$ and $\widetilde{q}_{i,\overline{j}}$ of the $Q_{i,\overline{j}}$ and $\widetilde{Q}_{i,\overline{j}}$ chiral multiplets are associated with the complex deformations of the affine $\widehat{A}_k$ singularity of the pp waves.

4.2. Ordinary CFT$_4$ models

Here we discuss how the above results can be extended to the case where, in addition to bi-fundamental matter, we also have fundamental matter carrying flavour charges. A class of such extensions is comprised of the four-dimensional $\mathcal{N} = 2$ conformal field theories based on *finite* ADE geometries. After a brief review of these models, we show how the vanishing of the corresponding beta function may be ensured.

It is recalled that for any Cartan matrix K_{ij} of finite ADE Lie algebras and a positive integer vector N_j, the linear combination M_i with

$$M_i = \sum_j K_{ij} N_j \tag{4.23}$$

is in general a positive integer vector. These positive integers M_i, which were null in affine models as in (4.10), are interpreted as the number of fundamental matter of the supersymmetric $U(N_i)$ quiver gauge theory.

Conformal field theories with fundamental matter require a non-trivial flavour symmetry G_{f}. This result is manifestly exhibited by the constraint equations (4.23), where the N_i integers are the orders of the gauge group factors $U(N_i) \subset \left[\otimes_{i=1}^{k-1} U(N_i)\right]$ and M_is are the orders of the $U(M_i)$ flavour global symmetries. In the following table, we list the basic content of an $\mathcal{N} = 2$ ordinary A_{k-1} CFT$_4$. The flavour symmetry is $G_{\mathrm{f}} = \left[\otimes_{i=1}^{k-1} U(M_i)\right]$, while the gauge group G_{g} is taken as a product of $U(N_i)$ factors with the condition $2N_i \geqslant N_{i-1} + N_{i+1}$ for $2 \leqslant i \leqslant k-2$ and $2N_1 \geqslant N_2$ and $2N_{k-1} \geqslant N_{k-2}$.

Gauge group G_{g}	$\left[\otimes_{i=1}^{k-1} U(N_i)\right]$	
Flavour group G_{f}	$U(2N_1 - N_2) \otimes U(2N_{k-1} - N_{k-2})$ $\otimes \left[\otimes_{i=2}^{k-2} U(2N_i - N_{i-1} - N_{i+1})\right]$	(4.24)
Bi-fundamental matter	$\oplus_{i=1}^{k-2} (N_i, \overline{N}_{i+1})$	
Fundamental matter	$\oplus_{i=1}^{k-1} (M_i N_i)$	

The scale invariance now fixes the number of fundamental matter M_i as follows: $M_1 = 2N_1 - N_2$, $M_{k-1} = 2N_{k-1} - N_{k-2}$ and $M_i = 2N_i - N_{i-1} - N_{i+1}$ for $2 \leqslant i \leqslant k-2$. These numbers satisfy manifestly the natural constraint equation

$$N_1 + N_{k-1} = M_1 + \cdots + M_{k-1}. \tag{4.25}$$

Actually this relation may be thought of as an equation classifying the kinds of critical models for the finite A_{k-1} category. The partitions of the number $N_1 + N_{k-1}$ give various kinds of finite A_{k-1} critical models with fundamental matter. Note that for the special case of a flavour symmetry group of type $U(M_2) \otimes U(M_{k-2})$ with M_2N_2 and $M_{k-2}N_{k-2}$ fundamental matter, the constraint equations for conformal invariance read

$$M_i = 0, \qquad \text{for} \quad i \neq 2, \quad k-2. \tag{4.26}$$

In the simplest case of this kind in which $M_2 = M_{k-2} = N$, it follows that the constraint equation for scale invariance is solved by

$$\begin{aligned} N_1 &= N, \\ N_i &= 2N, \qquad 2 \leqslant i \leqslant k-2, \\ N_{k-1} &= N. \end{aligned} \tag{4.27}$$

5. String and brane interpretations

Before studying the extension of the BMN correspondence between the various $\mathcal{N} = 2$ CFT_4s and superstrings in the Penrose limit of $AdS_5 \times S^5/\Gamma$ orbifolds, let us first give their interpretations in terms of D-branes living on ADE geometries.

5.1. String-theory analysis

In the Penrose limit of $AdS_5 \times S^5/\Gamma$ taken along the great circle of the S^5 fixed by the orbifold, the initial 32 supercharges are reduced by half. In this case, type IIB superstring theory on a pp-wave background with ADE geometry may be described by eight worldsheet scalars X^I and eight pairs of worldsheet Majorana fermions $\left(\vartheta_1^{\mathrm{A}}, \vartheta_2^{\mathrm{A}}\right)$. All of these fields are free but massive. In type IIB superstring theory, ϑ_1 and ϑ_2 have the same chirality, while the masses of the X^I scalars and the ϑ_i^{A} fermions, being equal by worldsheet supersymmetry, are effectively given by the RR 5-form field strength μ.

Moreover, the description of the Penrose limit of $AdS_5 \times S^5$ is based on $\mathbb{R}^4 \times \mathbb{R}^4$ rather than $\mathbb{R}^8$. This is manifested in the breaking of the eight-dimensional string light-cone $SO(8)$ symmetry rotating the eight worldsheet scalars X^I and their $\left(\vartheta_1^{\mathrm{A}}, \vartheta_2^{\mathrm{A}}\right)$ superpartners, as it is broken down to the $SO(4)_1 \otimes SO(4)_2$ homogeneous group of the $\mathbb{R}^4 \times \mathbb{R}^4$ space of coordinates (x^i, y^j). The eight scalars X^I and their $\left(\vartheta_1^{\mathrm{A}}, \vartheta_2^{\mathrm{A}}\right)$ superpartners therefore decompose as

$$\begin{aligned} &X^I \sim 8_v = (4_v, 1) + (1, 4_v) \\ &\vartheta^{\mathrm{A}} \sim 16_s = (4_s, 4_s), \qquad \Gamma_{11}\vartheta^{\mathrm{A}} = 0, \end{aligned} \tag{5.1}$$

where we have set $\vartheta^{\mathrm{A}} \equiv \left(\vartheta_1^{\mathrm{A}} + \mathrm{i}\vartheta_2^{\mathrm{A}}\right)/\sqrt{2}$ and the numbers appearing in the above relations stand for the dimensions of the representations. In these relations, the representation $(4_v, 1) + (1, 4_v)$ refers to (x^i, y^j), and, due to the homomorphism $SO(4) = SU(2) \otimes SU(2)$, each $SO(4)$ spinor representation 4_s can be read as $(2, 2)$. The second relation of (5.1) is solved as $\vartheta^{\mathrm{A}} \sim \left(\chi_+^{\alpha}, \xi_-^{\dot\alpha}\right)$. Here α and $\dot\alpha$ are spinor indices of $SU(2) \otimes SU(2)$ while the sub-indices carried by $\left(\chi_+^{\alpha}, \xi_-^{\dot\alpha}\right)$ refer to the chiralities with respect to $SO(4)$. These group multiplets can be further decomposed by working in the complex space $\mathbb{R}^4 \times \mathbb{C}^2$ instead of $\mathbb{R}^4 \times \mathbb{R}^4$. As such, the $SO(4)_2$ symmetry group rotating the four y^js is broken down to $U(2)$ and so the $SO(4)$ 4-vector is now viewed as a $U(2)$ 2-spinor. Similarly, the $SO(4)$ bi-spinor $(2, 2)$ is now reduced to $(2, +1) + (2, -1)$. A further breaking of $U(2)$ down to $U(1) \otimes U(1)$ leads to the following splitting of the worldsheet fields:

$$\begin{aligned} &X^I \to (x^i, y^j) \to (x^i, z^1, z^2), \\ &\vartheta^{\mathrm{A}} \to \left(\chi_+^{\alpha}, \xi_-^{\dot\alpha}\right) \to (\lambda^{\alpha}, \psi, \overline{\chi}), \end{aligned} \tag{5.2}$$

where $\chi_+^{\alpha} \equiv \lambda^{\alpha}$ and $\xi_-^{\dot\alpha} \equiv \xi^{\dot\alpha}$. It is noted that the spinor components $\xi^{\dot 1}$ and $\overline{\xi}^{\dot 2}$ transform in the same manner under the Abelian orbifold group $\mathbb{Z}_k$. It is therefore convenient to combine $\xi^{\dot 1}, \overline{\xi}^{\dot 2}$ and $\overline{\xi}^{\dot 1}, \xi^{\dot 2}$ into a Dirac spinor as $(\psi, \overline{\chi})$. This change of variables has been used in [22] to study type IIB superstrings propagating on pp waves over $\mathbb{R}^4/\mathbb{Z}_k$. We will also use it here.

5.1.1. Ordinary A_{k-1} geometry. Under blowing up of the $\mathbb{Z}_k$ orbifold singularity, the initial orbifold point is replaced by the covering of open sets $\mathcal{U}_i$, $1 \leqslant i \leqslant k$. The bosonic variables near the origin of $\mathbb{C}^3$ parameterized by $(\mathbf{x}, z_1, z_2)$ with $z_1 = z$, $z_2 = \frac{\zeta^{k+1}}{z}$ and their superpartners $(\boldsymbol{\lambda}^{\alpha}, \psi, \overline{\chi})$ thus get promoted to the sets $(\mathbf{x}_i, u_i, v_i)$ and $\left(\boldsymbol{\lambda}_i^{\alpha}, \psi_i, \overline{\chi}_i\right)$, respectively. Here the variables u_i and v_i are as in (3.9) and (3.10), and transform as in (3.12) whereas $\mathbf{x}_i \longrightarrow \mathbf{x}_i$

under $\mathbb{Z}_k$. The fields transform as

$$\boldsymbol{\lambda}_i^\alpha \longrightarrow \boldsymbol{\lambda}_i^\alpha, \qquad (\psi_i, \overline{\chi}_i) \longrightarrow (\omega^i \psi_i, \overline{\omega}^{i+1}\overline{\chi}_i). \tag{5.3}$$

In the mirror description, the analogue to the set $(\mathbf{x}_i, u_i, v_i)$ is given by $(\mathbf{x}_i, \tau_i)$, while the analogue to its superpartner $\left(\boldsymbol{\lambda}_i^\alpha, \psi_i, \overline{\chi}_i\right)$ is $\left(\boldsymbol{\lambda}_i^\alpha, \vartheta_i\right)$. Here τ_i are the mirror complex variables given by (3.19) and ϑ_i are their superpartners. Their transformations under the orbifold group are given by $\tau_i \longrightarrow \omega^{i-1}\tau_i$, cf (3.23), with a similar change for their fermionic partners. The different phases ω^n define the various sectors of the worldsheet fields sensitive to the $\mathbb{Z}_k$ orbifolding. Put differently, the fields get monodromies under $\mathbb{Z}_k$ transformations, resulting in the following twisted string sectors:

$$\tau_q(\sigma + 2\pi\alpha' p^+, \tau) = \omega_q^{q-1}\tau(\sigma, \tau), \qquad \vartheta_q(\sigma + 2\pi\alpha' p^+, \tau) = \omega^{q-1}\vartheta_q(\sigma, \tau). \tag{5.4}$$

The other worldsheet fields $\mathbf{x}_i$ and $\boldsymbol{\lambda}_i^\alpha$ remain periodic as usual

$$\mathbf{x}_q(\sigma + 2\pi\alpha' p^+, \tau) = \mathbf{x}_q(\sigma, \tau), \qquad \boldsymbol{\lambda}_q^\alpha(\sigma + 2\pi\alpha' p^+, \tau) = \boldsymbol{\lambda}_q^\alpha(\sigma, \tau). \tag{5.5}$$

The modes of these fields are given by

Field	$\mathbf{x}_q^i$	τ_q	$\boldsymbol{\lambda}_q^\alpha$	ϑ_q
Mode	a_n^i	$\tau_{n(q)}$	$\boldsymbol{\lambda}_n^\alpha$	$\vartheta_{n(q)}$

(5.6)

where $n(q) = n + \frac{n}{k}, n \in \mathbb{Z}$. This analysis extends naturally to the other orbifolds. The basic points are indicated here below in the cases of affine $\widehat{A}_k$ and affine $\widehat{D}_k$ geometries.

5.1.2. Affine $\widehat{A}_k$ geometry. In mirror affine $\widehat{A}_k$ geometry, there are $k+1$ sets of worldsheet variables $\left(\mathbf{x}_i, \boldsymbol{\lambda}_i^\alpha\right)$ and (τ_i, ϑ_i), $i = 0, \ldots, k$, associated with the $k+1$ nodes of the $\widehat{A}_k$ Dynkin diagram. The monodromy conditions of the A_k geometry in the qth twisted string sector, $q = 0, \ldots, k$, are given by

$$\begin{aligned} &\mathbf{x}_q(\sigma + 2\pi\alpha' p^+, \tau) = \mathbf{x}_q(\sigma, \tau), && \boldsymbol{\lambda}_q^\alpha(\sigma + 2\pi\alpha' p^+, \tau) = \boldsymbol{\lambda}_q^\alpha(\sigma, \tau), \\ &\tau_q(\sigma + 2\pi\alpha' p^+, \tau) = \omega^q\tau(\sigma, \tau), && \vartheta_q(\sigma + 2\pi\alpha' p^+, \tau) = \omega^q\vartheta_q(\sigma, \tau), \end{aligned} \tag{5.7}$$

while their oscillator modes are as in table (5.6).

5.1.3. Affine $\widehat{D}_k$ geometry. In the affine $\widehat{D}_k$ geometry, we have $k+1$ sets of worldsheet variables $\left(\mathbf{x}_i, \boldsymbol{\lambda}_i^\alpha\right)$ and (τ_i, ϑ_i), $i = 1, \ldots, k+1$, associated with the $k+1$ nodes of the affine $\widehat{D}_k$ Dynkin diagram. Since the dihedral group of $\widehat{D}_k$ geometry contains $\mathbb{Z}_{k-2}$ as an Abelian subsymmetry, the monodromy conditions of the $\widehat{D}_k$ geometry read

$$\begin{aligned} &\tau_q(\sigma + 2\pi\alpha' p^+, \tau) = \alpha_q\tau_q(\sigma, \tau), && q = 1, 2, \\ &\tau_q(\sigma + 2\pi\alpha' p^+, \tau) = \omega^{q-3}\tau_q(\sigma, \tau), && 3 \leqslant q \leqslant k-1, \\ &\tau_q(\sigma + 2\pi\alpha' p^+, \tau) = \beta_q\tau_q(\sigma, \tau), && q = k, k+1, \end{aligned} \tag{5.8}$$

and similarly for their fermionic partners. The modes are

Field	τ_1	τ_2	$\cdots$	τ_k	τ_{k+1}
Mode	$\tau_{n(\alpha_1)}$	$\tau_{n(\alpha_2)}$	$\cdots$	$\tau_{n(\alpha_k)}$	$\tau_{n(\alpha_{k+1})}$

(5.9)

In the special case where all α_qs are equal to 1, one recovers the usual periodic boundary conditions.

Having described the essentials on the string-theory side, we now turn to the field-theory interpretation of these models.

5.2. Field-theory analysis

The AdS/CFT correspondence connects the spectrum of type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ with $\mathcal{N} = 4U(N)$ SYM_4 (CFT_4) operators on the boundary $\partial(\mathrm{AdS}_5)$. It is also at the basis for other induced correspondences involving theories with a lower number of supercharges. Constructions based on taking Penrose limits and those using orbifolding by discrete groups Γ are concrete examples. In the previous subsection, we have studied the string-theory side. Here we want to explore the field-theory side. Both of them are needed for working out the extension of the BMN correspondence which will be considered in the next section.

We start by noting that in type IIB brane language, the engineering of $U(kN)$ SYM_4 theories can be achieved in different ways. The latter live in the worldvolume of D-branes which, according to the number of preserved supercharges and criticality, may involve various systems of branes. In particular, we have the following D-brane configurations:

(i) kN parallel D3-branes filling the four-dimensional spacetime and located at the origin of the transverse $\mathbb{R}^6$ space as one usually does in brane engineering of $\mathcal{N} = 4U(kN)$ SYM_4.

(ii) kN parallel D5-branes wrapping vanishing 2-cycles and filling the four-dimensional spacetime. These branes are located at the origin of the transverse $\mathbb{R}^4$ space but the resulting $U(kN)$ gauge theory has $\mathcal{N} = 2$ supersymmetry. Blowing up the vanishing spheres breaks the $U(kN)$ gauge symmetry down to $[\otimes_{i=1} U(N_i)]$, as already discussed. This is the usual system involved in the engineering of $\mathcal{N} = 2$ quiver QFT_4s based on *finite* Lie algebras.

(iii) One may also have systems involving both D3- and D5-branes. The D3-branes are located at the origin of the transverse $\mathbb{R}^6$ space, whereas the D5-branes are wrapped on vanishing 2-cycles located at the origin of the transverse space $\mathbb{R}^4 \subset \mathbb{R}^6$. This is the kind of brane system one has in the brane engineering of $\mathcal{N} = 2$ quiver CFT_4s classified by affine Kac–Moody algebras. In quiver Dynkin diagram representation, the affine node is associated with D3-branes and the others with D5-branes.

(iv) Parallel D7-branes wrapped on 4-cycles and located at a point on $\mathbb{R}^2 \subset \mathbb{R}^4 \subset \mathbb{R}^6$. In the case where the 4-cycles are complex surfaces of the Hirzebruch type, that is, spheres fibred over spheres, the resulting gauge theory has $\mathcal{N} = 1$ supersymmetry. However, here we will consider the D7-branes wrapped on 4-cycles with large volumes. They are needed in the brane engineering of $\mathcal{N} = 2$ CFT_4s based on either finite or indefinite Lie algebras.

5.2.1. Affine $\widehat{A}_k$ model. As noted before, this kind of $\mathcal{N} = 2$ quiver CFT_4 involves D3-branes and open strings stretching between them. The total number of D-branes is kN and may be partitioned into $k+1$ factors. $\mathcal{N} = 2$ conformal invariance therefore requires a $U(N)^{k+1}$ gauge symmetry with $k + 1$ bi-fundamentals interpreted as open strings stretching between them. A picture of the repartition of these branes is as in the following loop:

$$N \cdot \mathrm{D5} + \cdots + N \cdot \mathrm{D5} + N \cdot \mathrm{D5} + N \cdot \mathbf{D3} + N \cdot \mathrm{D5} + N \cdot \mathrm{D5} + \cdots + N \cdot \mathrm{D5}, \tag{5.10}$$

where the two sets of N D5-branes at the ends are identified. From this D-brane configuration, one sees that we have two kinds of open strings: those stretching between D3–D5, and open strings between D5–D5. In geometric engineering, this quiver is represented by (4.19) which is merely the Dynkin diagram of affine $\widehat{A}_k$. Each node, represented by a $N_i \times N_i$ matrix projector π_i which when restricted to the proper subspace is just the $N \times N$ identity matrix, corresponds to wrapping N D5-branes (N D3-branes for affine node) over 2-spheres. On this node thus lives an $\mathcal{N} = 2$ $U(N)$ gauge model. Links between the nodes describe bi-fundamental matter. The explicit field content of this $\mathcal{N} = 2$ CFT_4 is described presently.

Four-dimensional $\mathcal{N} = 2$ vector multiplet $\left(\mathcal{C}_i, \lambda_i^\alpha, \mathcal{A}_\mu^i\right)$. The gauge multiplet of the $\mathcal{N} = 2$ model, $(\mathcal{C}, \lambda^\alpha, \mathcal{A}_\mu)$, is reducible and splits as $\oplus_{i=1}^{k+1}\left(\mathcal{C}_i, \lambda_i^\alpha, \mathcal{A}_\mu^i\right)$. This property can also be derived from orbifold group theory analysis. Since the elements g of the orbifold group with an affine $\widehat{A}_k$ singularity are generated by $\Pi = \mathrm{diag}(1, \omega, \ldots, \omega^k)$, transformations of $(\mathcal{C}, \lambda^\alpha, \mathcal{A}_\mu)$ read

$$g\mathcal{A}_\mu g^{-1} = \mathcal{A}_\mu, \qquad g\lambda^\alpha g^{-1} = \lambda^\alpha, \qquad g\mathcal{C}g^{-1} = \mathcal{C}, \tag{5.11}$$

leaving the gauge-multiplet invariant. Moreover, as $g\Pi = \Pi g$, the solution to these equations may be expressed in terms of the projectors $\pi_i = |i\rangle\langle i|$:

$$V_0 = \sum_{j=1}^{k+1} V_j \pi_j. \tag{5.12}$$

Here V_0 stands for the massless $\mathcal{C}, \lambda_a^\alpha, \mathcal{A}_\mu$ fields and each component V_j of the expansion is a $N \times N$ Hermitian matrix. The first kV_js describe the gauge degrees of freedom in the jth subset of N D5-branes while the last one describes the gauge fields on the N D3-branes associated with the affine node.

Four-dimensional $\mathcal{N} = 2$ hypermultiplets $\mathcal{H}_{i,i+1}$ $= \left(\phi_{i,i+1}^\alpha, \psi_{i,i+1}, \overline{\chi}_{i,i+1}\right)$. Together with the $\mathcal{N} = 2$ gauge multiplets described above, there are also massless k hypermultiplets $\mathcal{H}_{i,i+1}$ describing the transverse positions of the $k + 1$ subsets of the N D-branes in the pp-wave geometry. These hypermultiplets form collectively a matrix $\mathcal{H}$ whose component fields ϕ^α, ψ and $\overline{\chi}$ transform under the orbifold group as

$$g\phi^\alpha g^{-1} = \omega\phi^\alpha, \qquad g\psi g^{-1} = \omega\psi, \qquad g\overline{\chi}g^{-1} = \overline{\omega\chi}. \tag{5.13}$$

The solution to these constraint equations reads

$$\mathcal{H} = \sum_{q=1}^{k+1} \mathcal{H}_{i,i+1} a_i^+, \tag{5.14}$$

where $\mathcal{H}$ transforms in the $(N_i, \overline{N}_{i+1})$ bi-fundamental representation of $U(N_i) \otimes U(N_{i+1})$ and $a_j^\pm$ are the usual creation and annihilation operators

$$\begin{aligned} &a_i^+ = |i+1\rangle\langle i|, \qquad a_j^- = |i\rangle\langle i+1|, \\ &a_i^- a_i^+ = \pi_i = |i\rangle\langle i|, \\ &a_i^+\pi_i = \pi_{i+1}a_i^+, \qquad a_i^+\pi_i = \pi_{i+1}a_i^+. \end{aligned} \tag{5.15}$$

This analysis can be extended naturally to the other $\mathcal{N} = 2$ affine $\widehat{\mathrm{DE}}$ CFT_4s. As an illustration, the example based on the series $\widehat{D}_k$ is discussed in the following.

5.2.2. Affine $\widehat{D}_k$ CFT_4. If we leave aside geometry, the field-theory properties of $\widehat{D}_k$ CFT_4s are quite similar to those we have described for the $\widehat{A}_k$ affine case. Here we have N D3-branes and $(2k-5)N$ D5-branes with open strings stretching between them. The D-brane configuration we have in this case is

$$N \cdot \mathrm{D3} + N \cdot \mathrm{D5} + 2(k-4)N \cdot \mathrm{D5} + N \cdot \mathrm{D5} + N \cdot \mathrm{D5}. \tag{5.16}$$

So we have $2N^2$ open strings between D3–D5 and open strings stretching between the remaining D5–D5 pairs. Some of these open strings transform in the bi-fundamental representations $(N, \overline{N})$, some in $(2N, 2\overline{N})$, some in $(2N, \overline{N})$, while some transform in

$(N, 2\overline{N})$. The transformation of the gauge multiplet is essentially the same as in (5.11), while transformations of hypermultiplets in the bi-fundamental representations involve

$$\phi^\alpha = \begin{pmatrix} 0 & 0 & \phi^\alpha_{1,3} & 0 & & & \cdots & & 0 \\ 0 & 0 & \phi^\alpha_{2,3} & 0 & & & & & 0 \\ 0 & 0 & 0 & \phi^\alpha_{3,4} & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & \phi^\alpha_{4,5} & 0 & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & & & & 0 & \phi^\alpha_{k-2,k-1} & 0 & 0 \\ 0 & & & & & 0 & 0 & \phi^\alpha_{k-1,k} & \phi^\alpha_{k-1,k+1} \\ 0 & & & & & 0 & 0 & 0 & 0 \\ 0 & & & \cdots & & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (5.17)$$

and likewise for their fermionic partners.

Similar results may also be derived for the exceptional affine series. These solutions can be obtained directly from the corresponding Dynkin diagrams.

Now we turn to $\mathcal{N} = 2$ CFT$_4$s with fundamental matter. We first consider $\mathcal{N} = 2$ CFT$_4$s based on ordinary A_{k-1}, after which we give the results for the other geometries.

5.2.3. Ordinary A_{k-1} model. Contrary to the affine case, there are no D3-branes in the ordinary case. The $\mathcal{N} = 2$ quiver CFT$_4$ based on finite A_{k-1} geometry, in particular, lives on the worldvolume of wrapped D5-branes on spheres of the deformed A_{k-1} singularity, but has in addition D7-branes needed to ensure conformal invariance. This is a gauge theory having in addition to the gauge group $\left[\otimes_{i=1}^{k-1} U_{\mathrm{g}}(N_i)\right]$, a $\left[\otimes_{i=1}^{k-1} U_{\mathrm{f}}(M_i)\right]$ flavour invariance dictated by the D7-branes wrapped on 4-cycles. The latter are supposed to have volumes big enough for the corresponding gauge symmetry to become a global gauge symmetry. The brane system describing the $\mathcal{N} = 2$ quiver CFT$_4$s we are interested in here is indicated in the following table:

Coordinate	x^+	x^-	x^1	x^2	x^3	x^4	y^1	y^2	y^3	y^4
kN D5-branes	$\surd$	$\surd$	$\surd$	$\surd$	$-$	$-$	$\surd$	$\surd$	$-$	$-$
kM D7-branes	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$-$	$-$

(5.18)

The coordinates x^+, x^-, x^1 and x^2 parameterize the worldvolume of the D5-branes wrapped on the compact spheres of the deformed singularity of $\mathbb{R}^4/\mathbb{Z}_k$. The latter is parameterized by the transverse coordinates (y^1, y^2, y^3, y^4). The other two transverse coordinates x^3 and x^4, which are interpreted as the scalar partners of the gauge fields on the D5-branes, describe the motion of the wrapped D-branes in the direction transverse to the orbifold $\mathbb{R}^4/\mathbb{Z}_k$. We also have D5–D5 and D5–D7 systems in the bi-fundamental and fundamental representations, respectively, of the gauge groups. These models can be engineered as follows.

- A set of kN D5-branes wrapped on 2-cycles in the deformed A_{k-1} singularity of the orbifold point in the transverse space. These branes are located at the origin of $\mathbb{R}^4 \subset \mathbb{R}^6$ and are partitioned as follows:

$$N_1 \cdot \mathrm{D5} + N_2 \cdot \mathrm{D5} + \cdots + N_{k-1} \cdot \mathrm{D5}, \qquad (5.19)$$

  with $kN = N_1 + N_2 + \cdots + N_{k-1}$. Strings emanating and ending on the same subset of N_i D5-branes are associated with the A^i_μ gauge fields of the supersymmetric multiplets.
- Between these wrapped D5-branes, there are $k-1$ open stings stretching and transforming in the bi-fundamentals of $U(N_i) \otimes U(N_{i+1})$. In geometric engineering of QFT$_4$, one may think about the subset of N_i D5-branes as corresponding to D5-branes wrapped N_i folds

on the ith 2-sphere of the A_{k-1} geometry (3.8). The resulting quantum field theory based on (5.19) is a gauge theory with a gauge group $\left[\otimes_{i=1}^{k-1} U(N_i)\right]$ and admits $\mathcal{N} = 2$ supersymmetry without full conformal invariance. To recover scale invariance, one has to add fundamental matter. This is achieved as follows.

- A set of kM D7-branes wrapped on 4-cycles with large volumes. These D7-branes are located at the origin of $\mathbb{R}^2/\mathbb{Z}_k \subset \mathbb{R}^4/\mathbb{Z}_k \subset \mathbb{R}^6/\mathbb{Z}_k$ and are partitioned as follows:

$$M_1 \cdot \mathrm{D7} + M_2 \cdot \mathrm{D7} + \cdots + M_{k-1} \cdot \mathrm{D7}, \tag{5.20}$$

  where $kM = M_1 + M_2 + \cdots + M_{k-1}$ and M_is are as in (4.24).
- On each subset of N_i wrapped D5-branes emanate M_i open strings in the fundamental representation of $U(N_i)$ and ending on the M_i D7-branes. These open strings are needed to ensure conformal invariance.

Massless four-dimensional $\mathcal{N} = 2$ vector multiplets, $\mathcal{V} = \left(\mathcal{C}, \lambda_a^\alpha, \mathcal{A}_\mu\right)$. Under transformations g generated by the element $\Pi = \mathrm{diag}(1, \omega, \ldots, \omega^{k-1})$ of the orbifold group $\mathbb{Z}_k$, the $\mathcal{N} = 2$ massless gauge multiplet is invariant, that is,

$$g\mathcal{A}_\mu g^{-1} = \mathcal{A}_\mu, \qquad g\lambda^\alpha g^{-1} = \lambda^\alpha, \qquad g\mathcal{C}g^{-1} = \mathcal{C}. \tag{5.21}$$

The solution of these equations is $V = \sum_{j=1}^{k-1} V_j \pi_j$, where each component V_j of the expansion is a $N_j \times N_j$ Hermitian matrix. Along with these massless representations, there are also some massive $\mathcal{N} = 2$ gauge multiplets $W_{ij} = \left(\mathcal{C}_{ij}, \lambda_{ij}^\alpha, \mathcal{A}_\mu^{ij}\right)$ associated with the broken generator of the original $U(N_1 + N_k)$ group down to $[\otimes_i U(N_i)]$. These states break the conformal invariance and will not be discussed here.

Four-dimensional $\mathcal{N} = 2$ bi-fundamental matter, $\mathcal{H}_{i,i+1} = \left(\phi_{i,i+1}^\alpha, \psi_{i,i+1}, \overline{\chi}_{i,i+1}\right)$. Along with the $V_{i,j}$ gauge multiplets described above, there are also massless hypermultiplets $\mathcal{H}_{i,i\pm1}$ in the bi-fundamental representations of the $U(N_i) \otimes U(N_{i+1})$ gauge subgroups of $\left[\otimes_{j=1}^{k-1} SU(N_j)\right]$ describing the positions of the sets of N_i D3-branes, $i = 1, \ldots, k$, in the pp-wave background

$$\mathcal{H}_{1,\overline{2}}, \mathcal{H}_{2,\overline{3}}, \ldots, \mathcal{H}_{k-1,\overline{k}}. \tag{5.22}$$

Each hypermultiplet $\mathcal{H}_{i,\overline{j}}$ contains two $\mathcal{N} = 1$ chiral multiplets, which we denote as $q_{i,\overline{j}}$ and $\widetilde{q}_{i,\overline{j}}$, and whose charges under the $U_R(1)$ subsymmetry are $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. These matter fields transform under the orbifold group as in (5.13).

Four-dimensional $\mathcal{N} = 2$ fundamental matter, $\mathcal{H}_i = \left(\phi_i^\alpha, \psi_i, \overline{\chi}_i\right)$. The fundamental matter $\left\{\mathcal{H}_i^{a_i} \equiv (N_i, M_a)\right\}$, denoted also as $\left(q_i^{a_i}, \widetilde{q}_i^{a_i}\right)$ in terms of $\mathcal{N} = 1$ chiral multiplets, transforms in the (N_i, M_i) representation of the $U(N_i) \otimes U(M_i)$ subsymmetries of the A_{k-1} CFT$_4$. These multiplets carry no orbifold charge.

This analysis can be extended naturally to the $\mathcal{N} = 2$ CFT$_4$ based on finite or indefinite Lie algebras.

6. Operator/string-state correspondence in $\mathcal{N} = 2$ CFT$_4$

With all the above tools at hand, we are now ready to write down the extension of the BMN correspondence between type IIB superstring theory on pp waves with ADE orbifold geometries and $\mathcal{N} = 2$ CFT$_4$s. A path to obtain this extension is to use the following: (a) the original BMN proposal for $\mathcal{N} = 4U(|\Gamma|N)$ SYM$_4$ and type IIB superstring theory on a Penrose limit of AdS$_5 \times S^5$; (b) reduction of $\mathcal{N} = 4U(|\Gamma|N)$ SYM$_4$ down to $\mathcal{N} = 2$ quiver gauge CFT$_4$s according to picture (1.1); (c) ADE orbifolds of the Penrose limit of AdS$_5 \times S^5$ studied in section 3. In what follows, we first build gauge-invariant field operators, then we give the corresponding string states. We conclude this section by commenting on certain specific states of the various classes of $\mathcal{N} = 2$ CFT$_4$s.

6.1. Gauge-invariant field operators

Following [10], there is a correspondence between type IIB superstring states on pp-wave geometries with quantized energy $E = \Delta - J = n$, with n a positive integer, and gauge-invariant states of $\mathcal{N} = 4U(N)$ SYM$_4$. This relation applies to orbifolds as well, except that now n is a positive *half-integer* since the scalars in the matter hypermultiplets carry positive half-integer charges J.

On the field-theory side, orbifolding $\mathcal{N} = 4U(N)$ SYM$_4$ by a discrete group Γ promotes the gauge group to $U(|\Gamma|N)$. However, deformations of the orbifold singularity break the $\mathcal{N} = 4U(|\Gamma|N)$ SYM$_4$ down to $\mathcal{N} = 2$ quiver gauge CFT$_4$s with gauge group $[\otimes_i U(N_i)]$. The initial $\mathcal{N} = 4$ $U(|\Gamma|N)$ vector multiplet $\left(0^6, 1, \frac{1}{2}^4\right) \otimes [U(|\Gamma|N)]_{\mathrm{adj}}$, where $[U(N_i)]_{\mathrm{adj}} = N_i \otimes \overline{N}_i$, decomposes into two kinds of representations: (a) $\mathcal{N} = 2$ vector multiplets $\left(0^2, 1, \frac{1}{2}^2\right)_i \otimes [U(N_i)]_{\mathrm{adj}}$ which in terms of the component fields read

$$\sum_i \left(\mathcal{C}_i, A^i_\mu, \lambda^i_{1,2}\right) \otimes [U(N_i)]_{\mathrm{adj}}, \tag{6.1}$$

where $\mathcal{C}_i$ is the usual complex scalar gauge field with superpartners A^i_μ and $\lambda^i_{1,2}$; (b) hypermultiplets in bi-fundamental representations of the gauge group with field content given by

$$\left[\sum_{i,j} \left(0^4, \frac{1}{2}^2\right)_{ij} \otimes (N_i \otimes \overline{N}_j)\right] \oplus \left[\sum_i M_i \left(0^4, \frac{1}{2}^2\right)_i \otimes N_i\right], \tag{6.2}$$

with the second term describing fundamental matter. It goes beyond the residue one gets from the reduction $\mathcal{N} = 4U(|\Gamma|N)$ SYM$_4$ and has been introduced here to recover conformal invariance for the case of singularities classified by finite ADE Lie algebras.

In $\mathcal{N} = 1$ formalism, the $\mathcal{N} = 2$ vector multiplet $\left(0^2, 1, \frac{1}{2}^2\right)_i$ is represented by a $\mathcal{N} = 1$ gauge multiplet $\left(\frac{1}{2}, 1\right)$ and a chiral multiplet $\left(0^2, \frac{1}{2}\right) \equiv C_i$ in the adjoint representation of $U(N_i)$. We shall think about these adjoint supermultiplets as representing the nodes of a quiver diagram as

$$\cdots \quad O^{\overline{N}_{i-1}}_{N_{i-1}} \qquad O^{\overline{N}_i}_{N_i} \qquad O^{\overline{N}_{i+1}}_{N_{i+1}} \quad \cdots. \tag{6.3}$$

Similarly, hypermultiplets $\left(0^4, \frac{1}{2}^2\right)_{ij}$ in bi-fundamental representations of $U(N_i) \otimes U(N_i)$ decompose into two kinds of chiral multiplets, $\left(0^2, \frac{1}{2}\right)_{ij}$ and $\left(0^2, \frac{1}{2}\right)_{ji}$. The two chiral superfields, which we denote as Φ_{ij} and Φ_{ji} with $i < j$, transform as upper components of two $SU_R(2)$ iso-doublets and have a charge $J = \frac{1}{2}$. They are in the $(N_i, \overline{N}_j)$ and $(\overline{N}_j, N_i)$ bi-fundamental representations of $U(N_i) \otimes U(N_i)$, respectively, and may be thought of as the links between nodes

$$\cdots \quad \leftrightarrows \quad O^{\overline{N}_{i-1}}_{N_{i-1}} \quad \leftrightarrows \quad O^{\overline{N}_i}_{N_i} \quad \leftrightarrows \quad O^{\overline{N}_{i+1}}_{N_{i+1}} \quad \leftrightarrows \quad \cdots. \tag{6.4}$$

The arrow $\longrightarrow$ from i to j refers to Φ_{ij} and the opposite to Φ_{ji}. Besides $U(N_i)$, fundamental hypermultiplets $\left(0^4, \frac{1}{2}^2\right)_i$ of (6.2) transform moreover under the flavour group $U(M_i)$ and are decomposed in two chiral multiplets $Q^\pm_i$ with charge $J_i = \frac{1}{2}$. The Q^-_i and Q^+_i multiplets transform as $(N_i, \overline{M}_i)$ and $(\overline{N}_i, M_i)$ of $U(N_i) \otimes U_{\mathrm{f}}(M_i)$. The scalar fields associated with the Φ_{ij} and $Q^\pm_i$ multiplets will be denoted as ϕ_{ij} and $q^\pm_i$.

Using these fields, we can build gauge-invariant field operators involving a single trace. The simplest ones are given by

$$\mathbb{O}_i^{(0)} = \mathrm{Tr}\big[\mathcal{C}_i^J\big], \qquad \mathbb{O}_j^{(1)} = \mathrm{Tr}\left(\sum_i \phi_{ji}\mathcal{C}_i^{J_i}\phi_{ij}\mathcal{C}_j^{J_j}\right). \tag{6.5}$$

The first field operator has $\Delta - J = 0$, while the second one has $\Delta - J = 1$. Note that introducing the Π_i projectors on the ith set of N_i D-branes, one can put the previous operators in compact form as

$$\mathbb{O}_0 = \sum_i \Pi_i \mathbb{O}_i^{(0)}, \qquad \mathbb{O}_1 = \sum_i \Pi_i \mathbb{O}_i^{(1)}, \tag{6.6}$$

with $\Delta - J = 0$ and $\Delta - J = 1$, respectively. Along with these gauge-invariant operators, one may also have gauge-invariant field operators that are not singlets with respect to the flavour symmetry. This concerns operator involving fundamental matter such as the operators

$$\mathbb{M}^{(1)}_{\alpha_i\overline{\beta}_i} = q^+_{\alpha i}\mathcal{C}_i^{J_i}q^-_{i\overline{\beta}}, \qquad \mathbb{M}^{(2)}_{\alpha_i\overline{\beta}_i} = q^+_{\alpha i}\mathcal{C}_i^{J_i}\phi_{ij}\mathcal{C}_j^{J_j}\phi_{ji}q^-_{i\overline{\beta}}, \tag{6.7}$$

with $\Delta - J = 1$ and $\Delta - J = 2$ transforming in the adjoint representation of the flavour symmetry $U(M_i)$. These relations are generalized straightforwardly.

6.2. Correspondence

On the string-theory side, we have both closed- and open-string states. The open sector is associated with fundamental matter. Before giving these states, recall that orbifolding by Γ induces twisted string sectors with $|\Gamma|-1$ closed-string ground states $|0, p^+\rangle_i^{\mathrm{lc}}$. The remaining state corresponds to the untwisted case, though we shall treat them collectively here below. Since these vacua have zero energy, it is natural to identify them with the following chiral operators:

$$\frac{1}{N_i^{J/2}\sqrt{J_i}}\mathbb{O}_i^{(0)} \longleftrightarrow |0, p^+\rangle_i^{\mathrm{lc}}, \qquad i = 1, \ldots, |\Gamma|. \tag{6.8}$$

As on the field-theory side, here also one may use the Π_i projectors on the ith block of D-branes to reformulate this correspondence in a compact form. We thus have

$$|0, p^+\rangle_{\mathrm{lc}} = \sum_i |0, p^+\rangle_i^{\mathrm{lc}}\Pi_i, \qquad |0, p^+\rangle_i^{\mathrm{lc}} = \Pi_i|0, p^+\rangle_{\mathrm{lc}}, \tag{6.9}$$

resulting in the correspondence

$$\mathbb{O}_0 \longleftrightarrow |0, p^+\rangle_{\mathrm{lc}}, \tag{6.10}$$

where $\mathbb{O}_0$ is as in (6.6). This relation formally resembles the original BMN proposal for the vacuum in type IIB superstring theory on pp waves of $\mathrm{AdS}_5 \times S^5$. One can therefore mimic their approach to write down the excited closed-string states. Note that one may also write the light-cone gauge Hamiltonian H_{lc} using eigenvalues as

$$H_{\mathrm{lc}} = \sum_i (\Delta_i - J_i)\Pi_i, \qquad \Delta_i - J_i > 0. \tag{6.11}$$

Each string oscillator corresponds to the insertion of a $\Delta - J = 1$ gauge-invariant field operator, summing over all positions with an independent phase, according to the rule

$$\begin{aligned} &a_{n_i}^{\dagger\mu} \to D_\mu\mathcal{C}_i, && \mu = 1, \ldots, 4, \quad i = 1, 2, \ldots, \\ &b^\dagger_{ij} \to \phi_{ij}, && i < j, \end{aligned}$$

$$\begin{aligned}
\widetilde{b}^{\dagger}_{ij} &\to \phi_{ji}, \qquad i < j,\\
\lambda^{\dagger\alpha} &\to \lambda^{\alpha}_{J=1/2},\\
\psi_{ij} &\to \psi_{J=1/2},
\end{aligned} \tag{6.12}$$

where D_μ is the gauge-covariant derivative. For states with $\Delta_i - J_i = 1$, for instance, we have the following two examples of correspondence for $\mathcal{N} = 2$ CFT_4 closed-string states:

$$\begin{aligned}
\frac{1}{N_i^{J_i/2}\sqrt{J_i}}\,\mathrm{Tr}\big[\mathcal{C}_i^{J_i} D_\mu \mathcal{C}_i\big] &\longleftrightarrow a_i^{\mu\dagger}\Pi_i|0, p^+\rangle_{\mathrm{lc}},\\
\frac{1}{N_i^{J_i/2}\sqrt{J_i}}\,\mathrm{Tr}\big[\mathcal{C}_i^{J_i}\lambda_i^{\alpha}\big] &\longleftrightarrow \lambda_0^{\alpha+}\Pi_i|0, p^+\rangle_{\mathrm{lc}},
\end{aligned} \tag{6.13}$$

where $\lambda_0^{\alpha+}$ is a fermionic zero mode with $J = 1/2$. For states with $\Delta_i - J_i = 2$, we have, for example,

$$\frac{1}{N_i^{J_i/2}\sqrt{J_i}}\sum_{l_i=1}^{J_i}\mathrm{Tr}\big[\mathcal{C}_i^{l_i}(D_\mu\mathcal{C}_i)\mathcal{C}_i^{(J_i-l_i)}D_\nu\mathcal{C}_i\big]\exp\left(\frac{2\mathrm{i}\pi l_i}{J_i}\right) \longleftrightarrow a_{n_i}^{\mu+}a_{-n_i}^{\nu+}\Pi_i|0, p^+\rangle_{\mathrm{lc}}, \tag{6.14}$$

where n_i is a positive integer for all i.

Together with these closed-string states, there are states involving open strings stretching between D-branes. These correspond to the insertion of matter fields ϕ_{ij} having no J_{ij} charge. For states with $\Delta - J = 2$ or 3, we have among other examples the following typical gauge-invariant field operators:

$$\phi_{ji}\mathcal{C}_i^{J_i}\phi_{ij}\mathcal{C}_j^{J_j} \qquad \text{and} \qquad \phi_{ji}\mathcal{C}_i^{J_i}\phi_{ik}\mathcal{C}_k^{J_k}\phi_{kj}\mathcal{C}_j^{J_j}. \tag{6.15}$$

They correspond to the states

$$\widetilde{b}^{+}_{n_{ij}}b^{+}_{-n_{ij}}\Pi_j|0, p^+\rangle_{\mathrm{lc}} \qquad \text{and} \qquad \widetilde{b}^{+}_{n_{ji}}b^{+}_{-n_{ik}}b^{+}_{-n_{kj}}\Pi_j|0, p^+\rangle_{\mathrm{lc}}, \tag{6.16}$$

respectively, where $n_{ij} = n - (j - i)/|\Gamma|$ and $n_{ij} + n_{jk} + \cdots + n_{li} = 0$. It is noted that the first closed-string state is built out of two open strings stretching between the ith and jth D-branes and constituting a loop emanating and ending on the jth D-brane. The second state involves three open strings forming a loop. Note also that on the field-theory side, open-string states stretching between the ith and kth wrapped D5-branes are associated with $\mathcal{C}_i^{J_i}\phi_{ik}\mathcal{C}_k^{J_k}$. The composition of two open strings may give either an open string as for $\mathcal{C}_j^{J_j}\phi_{ji}\mathcal{C}_i^{J_i}\phi_{ik}\mathcal{C}_k^{J_k}$ or a closed-string state as for the gauge-invariant operator $\phi_{ji}\mathcal{C}_i^{J_i}\phi_{ij}\mathcal{C}_j^{J_j}$. In this D-brane engineering of QFT, bi-fundamental matter corresponds to open strings stretching between D3–D5 and D5–D5 branes. However, one may still build gauge-invariant field operators that are dual to open strings stretching between D7-branes. This requires the introduction of fundamental matter as in the following examples:

$$q_{\alpha i}\mathcal{C}_i^{J_i}\widetilde{q}_{i\overline{\alpha}} \qquad \text{and} \qquad q_{\alpha i}\mathcal{C}_i^{J_i}\phi_{ij}\mathcal{C}_j^{J_j}\phi_{ji}\widetilde{q}_{i\overline{\alpha}}. \tag{6.17}$$

These gauge-invariant field states involve two quarks in the fundamental representation of the flavour symmetry $U(M_i)$ and have a light-cone energy E_{lc} equal to 2 and 3, respectively.

6.3. *Specific states*

Here we want to describe some particular gauge-invariant field operators that are specific to the various $\mathcal{N} = 2$ ADE quiver CFT_4s. These states concern both closed and open strings, and are related to the topology of ADE Dynkin diagrams.

6.3.1. $\mathcal{N} = 2A_{k-1}$ quiver CFT$_4$s. To fix the ideas, we consider here the example of $\mathcal{N} = 2$ quiver gauge SYM$_4$ based on an ordinary A_{k-1} singularity. This is in general not a conformal $\mathcal{N} = 2$ QFT$_4$ but may be converted into one by adding fundamental matter. In this case, the group symmetry of the CFT is of the form $G = G_{\mathrm{g}} \otimes G_{\mathrm{f}}$. A simple example corresponds to a gauge group $G_{\mathrm{g}} = U(N)^{k-1}$ and $G_{\mathrm{f}} = U(M_1) \otimes U(M_{k-1})$ given by $U(N)^2$. This is then a supersymmetric conformal model with $M_1 + M_{k-1} = N + N$ fundamental matter, respectively, in the N_1 and N_{k-1} representations of the $U(N_1) \otimes U(N_{k-1})$ gauge symmetry, which in the present example is also equal to $U(N)^2$. A gauge-invariant field operator of this model, having light-cone energy $k + 1$ with two quarks at the ends, is shown here below

$$\mathbb{M}_{\alpha,\overline{\alpha}} = \mathrm{Tr}\big(q^{-}_{\alpha 1}\phi_{01}\mathcal{C}_1^{J_1}\phi_{12}\mathcal{C}_2^{J_2}\cdots\phi_{i-1i}\mathcal{C}_i^{J_i}\phi_{ii+1}\cdots\mathcal{C}_{k-1}^{J_k}q^{+}_{k-1\overline{\alpha}}\big), \tag{6.18}$$

where the trace is over the gauge group. The introduction of the two quarks ensures gauge invariance. This field operator transforms in the bi-fundamental representation of the flavour group $U(N_1) \otimes U(N_{k-1})$, and it corresponds to an open string stretching between the sets of M_1 and M_{k-1} D7-branes.

6.3.2. $\mathcal{N} = 2$ affine $\widehat{A}_k$ quiver CFT$_4$s. An $\mathcal{N} = 2$ quiver gauge theory based on affine $\widehat{A}_k$ is a CFT$_4$ without the need of introducing fundamental matter. As affine $\widehat{A}_k$ has a Dynkin diagram represented by a loop (4.19), the latter may be viewed as a closed string of energy $k + 1$ built out of the open strings stretching between the D-branes. Adjunction of fundamental matter in this class of CFTs is possible, but leads to interacting open strings.

Let us discuss hereafter the case without fundamental matter. In this case, affine $\widehat{A}_k\mathcal{N} = 2$ quiver CFT$_4$s have gauge group $U(N)^{k+1}$ and bi-fundamental matter. Closed-string state of light-cone energy $k + 1$ involves $k - 1$ D5–D5 open strings and two D3–D5 open strings. In the large-N limit, the corresponding gauge-invariant field operator reads

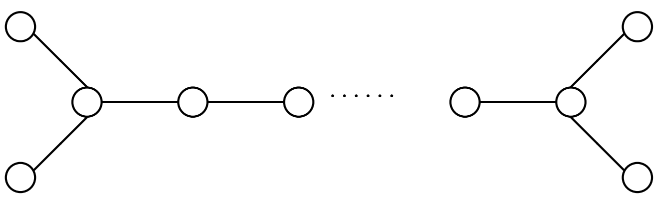
$$\mathrm{Tr}\big(\phi_{01}\mathcal{C}_1^{J_1}\phi_{12}\mathcal{C}_2^{J_2}\cdots\phi_{i-1i}\mathcal{C}_i^{J_i}\phi_{ii+1}\cdots\mathcal{C}_k^{J_k}\phi_{k0}\mathcal{C}_0^{J_0}\big). \tag{6.19}$$

Here ϕ_{k0} and ϕ_{01} represent open strings stretching between D5–D3 branes, while the remaining fields ϕ_{ii+1} correspond to open strings stretching between D5–D5 branes. As above, J_is are all of order $\sqrt{N}$.

Adding fundamental matter while keeping scale invariance requires more general group symmetries. In geometric engineering, such a CFT is described by diagrams with open topologies involving trivalent vertices as in [31]. On the string-theory side, these topologies may be interpreted as describing interacting open strings with quarks as external legs.

6.3.3. $\mathcal{N} = 2$ affine $\widehat{\mathrm{DE}}$ quiver CFT$_4$s. One may also build the analogue of the previous gauge-invariant field operators for the other classes of $\mathcal{N} = 2$ CFT$_4$s. For affine $\widehat{D}_k$ series with gauge group $G_{\mathrm{g}} = U(N)^4 \otimes U(2N)^{k-3}$, the situation is interesting as it involves two 3-vertices as indicated in the associated Dynkin diagram:

$\widehat{D}_k$:



(6.20)

A direct way to obtain gauge-invariant field operators is to use the following representation: (a) vertices with two legs are associated with the adjoint representation $N_i \otimes \overline{N}_i$ of the corresponding gauge group $U(N_i)$ and are denoted as

$$\begin{array}{ccccccccc} \cdots & \longleftarrow & N_{i-1} & \longrightarrow & \overline{N}_i & \longleftarrow & N_{i+1} & \longrightarrow & \cdots \\ \cdots & \longrightarrow & \overline{N}_{i-1} & \longleftarrow & N_i & \longrightarrow & \overline{N}_{i+1} & \longleftarrow & \cdots \end{array} \tag{6.21}$$

where an arrow from N_i to $\overline{N}_{i+1}$ refers to bi-fundamental matter $\phi_{i,i+1}$, while an arrow from N_{i+1} to $\overline{N}_i$ refers to $\phi_{i+1,i}$. (b) The three-vertices, cf (6.20), are represented by

$$\begin{array}{ccccccccc} & & & & & \longleftarrow & N_k \\ & & & & & \longleftarrow & N_{k-1} \\ \cdots & \longleftarrow & N_{k-3} & \longrightarrow & \overline{N}_{k-2} & & \\ \cdots & \longrightarrow & \overline{N}_{k-3} & \longleftarrow & N_{k-2} & & \\ & & & & & \longrightarrow & \overline{N}_{k-1} \\ & & & & & \longrightarrow & \overline{N}_k . \end{array} \tag{6.22}$$

The use of arrows is quite similar to the situation for 2-vertices. Using this representation, one can write down the analogue of (6.19). We find the following gauge-invariant field operator for affine $\widehat{D}_k$ CFT$_4$s:

$$\begin{aligned} \mathrm{Tr}\Big(\big[\mathcal{C}_0^{J_0}\phi_{02}\mathcal{C}_2^{J_2} + \mathcal{C}_1^{J_1}\phi_{12}\mathcal{C}_2^{J_2'}\big] \times \phi_{23}\mathcal{C}_3^{J_3}\phi_{34}\cdots\phi_{i-1i}\mathcal{C}_i^{J_i}\phi_{ii+1}\cdots\phi_{k-3,k-2} \\ \times \big[\mathcal{C}_{k-2}^{J_{k-2}}\phi_{k-2,k-1}\mathcal{C}_{k-1}^{J_{k-1}} + \mathcal{C}_{k-2}^{J'_{k-2}}\phi_{k-2,k}\mathcal{C}_k^{J_k}\big]\Big). \end{aligned} \tag{6.23}$$

Open-string configurations may also be considered here and one has just to add fundamental matter by keeping gauge invariance. The results we derived here can be extended naturally to the remaining classes of $\mathcal{N} = 2$ CFT$_4$s [35, 36].

7. Conclusion

In this paper, we have studied the extension of the BMN proposal to type IIB superstring theory on pp-wave orbifolds preserving 16 supercharges. In particular, we have considered a large class of models describing certain limits of type IIB superstrings propagating on pp wave with ADE geometries. Based initially on closed strings only, we have developed an analysis for the two classes involving ordinary ADE singularities and affine $\widehat{\mathrm{ADE}}$ elliptic geometries. These models are dual to $\mathcal{N} = 2$ CFT$_4$ quiver theories classified, respectively, by finite ADE Lie algebras and affine $\widehat{\mathrm{ADE}}$ Kac–Moody algebras. Concerning the finite ADE quiver models, we have shown that conformal invariance requires the incorporation of fundamental matter being linked to open strings. Our main results in this paper may be summarized as follows. First, we have reviewed type IIB superstring theory on pp-wave orbifolds, in particular for orbifold groups $\Gamma \subset SU(2)$, itself contained in the R-symmetry $SU(4)$ of $\mathcal{N} = 4$ SYM$_4$. Then we have given the explicit form of the corresponding pp-wave metrics for both ordinary and affine geometries. After that we have studied the various classes of $\mathcal{N} = 2$ CFT$_4$s classified by finite and ordinary ADE Dynkin diagrams of Lie algebras. Finally, we have derived the appropriate extension of a BMN proposal for these models. In particular, we have given the correspondence rule between leading closed-string states and gauge-invariant operators in the $\mathcal{N} = 2$ superconformal quiver theories.

Our work opens up for further studies. One interesting problem is to complete this analysis by considering $\mathcal{N} = 2$ CFT$_4$ quiver models classified by indefinite Lie algebras [35, 36, 40–42]. In particular, it is interesting to write down the pp-wave geometries corresponding to such models. A natural question concerns the extension of this work to the case of pp-wave orbifolds with eight or four supercharges. We hope to report elsewhere on these open questions.

Acknowledgments

AB is very grateful to A Sebbar for discussion, hospitality and scientific help. JR thanks C Cummins for discussions. This work is supported in part by the program Protars III, CNRST D12/25, Rabat.

Appendix A. Non-Abelian and affine orbifolds

So far, we have worked out explicit examples of pp waves with Abelian orbifold geometry. Here we study their non-Abelian analogues by first considering the ordinary orbifold geometries and subsequently their affine counterparts.

A.1. Non-Abelian orbifolds

Since there are two basic classes of ADE singularities that one encounters in the study of type II superstrings on singular Calabi–Yau manifolds, namely the ordinary singularities associated with ordinary ADE Lie algebras and the affine ones associated with the $\widehat{\mathrm{ADE}}$ affine Kac–Moody extensions, we will divide our discussion on pp-wave backgrounds into two parts[7]. The first one deals with pp-wave geometries in connection with ordinary ADE Lie algebras and the second part concerns the geometries in connection with the affine $\widehat{\mathrm{ADE}}$ Lie algebras.

A.1.1. Ordinary DE pp waves. The DE singularities of the ALE space leading to four-dimensional $\mathcal{N} = 2$ models are described by the equations appearing in table (2.25). The analysis of the blow-up of these singularities goes along the same lines as in the Abelian case A_{k-1}, except now the non-Abelian property of the orbifold groups makes the analysis a bit tedious. We will give some details regarding both the blow-ups and the mirrors of the ordinary D_k and E_6 geometries. The results for E_7 and E_8 geometries follow from a similar analysis and will be omitted.

D_k pp-wave geometry. Starting from (3.1) and using the equation defining D_k singularity, which allow to express the z_1 variable in terms of z_2 and ζ as

$$z_1 = \sqrt{\zeta^{k-1} - z_2^2\zeta} \tag{A.1}$$

and the differential $\mathrm{d}z_1$ in terms of $\mathrm{d}z_2$ and $\mathrm{d}\zeta$ as

$$\mathrm{d}z_1 = \frac{1}{2\left(\zeta^{k-1} - z_2^2\zeta\right)^{\frac{1}{2}}}\left[2z_2\zeta\,\mathrm{d}z_2 + \left((k-1)\zeta^{k-2} - z_2^2\zeta\right)\mathrm{d}\zeta\right], \tag{A.2}$$

one can work out the metric of the pp-wave background near the orbifold point with D_k singularity. The result is given by

$$\mathrm{d}s^2|_{D_k} = -4\,\mathrm{d}x^+\,\mathrm{d}x^- - \mu^2\left(\mathbf{x}^2 + \left|\zeta^{k-1} - z_2^2\zeta\right|^2 + |z_2|^2\right)(\mathrm{d}x^+)^2 + \mathrm{d}\mathbf{x}^2$$
$$+ \left(1 + \left|\frac{2z_2\zeta}{2\left(\zeta^{k-1} - z_2^2\zeta\right)^{\frac{1}{2}}}\right|^2\right)|\mathrm{d}z_2|^2 + \left|\frac{(k-1)\zeta^{k-2} - z_2^2\zeta}{2\left(\zeta^{k-1} - z_2^2\zeta\right)^{\frac{1}{2}}}\right|^2|\mathrm{d}\zeta|^2$$

[7] For an extension to geometries involving indefinite Lie algebras, see [35, 36].

$$+\frac{2z_2\zeta}{2\left|\zeta^{k-1}-z_2^2\zeta\right|}\left((k-1)\overline{\zeta}^{k-2}-\overline{z}_2^2\overline{\zeta}\right)\mathrm{d}z_2\,\mathrm{d}\overline{\zeta}$$
$$+\frac{2\overline{z}_2\overline{\zeta}}{2\left|\zeta^{k-1}-z_2^2\zeta\right|}\left((k-1)\zeta^{k-2}-z_2^2\zeta\right)\mathrm{d}\overline{z}_2\,\mathrm{d}\zeta, \tag{A.3}$$

which is a singular metric at the origin $z_2 = \zeta = 0$. As usual, this can be resolved either by Kähler or complex deformations. We will consider both cases to which we shall refer hereafter as blown-up D_k pp-wave and mirror D_k pp-wave geometries, respectively.

Blown-up D_k pp-wave geometry. In the blow-up of the ordinary D_k singularity, one introduces a basis of k complex three-dimensional open sets $\mathcal{U}_i = \{(u_i, v_i, w_i), 1 \leqslant i \leqslant k\}$ glued together as [38]

$$\begin{aligned} u_i &= u_{i+1}v_{i+1}w_{i+1}, \\ v_i &= u_{i+1}, \qquad 1 \leqslant i \leqslant k-4, \\ w_i v_{i+1} &= w_{k-3}v_{k-2} = w_{k-2}v_{k-1} = w_k v_{k-1} = 1, \end{aligned} \tag{A.4}$$

supplemented by

$$\begin{aligned} u_{k-3} &= u_{k-2}v_{k-2}^2 w_{k-2}, \qquad & v_{k-3} &= u_{k-2}v_{k-2}, \\ u_{k-2}v_k &= v_{k-1}w_{k-1}v_k = 1, \qquad & v_{k-2} &= u_{k-2} = u_k v_k. \end{aligned} \tag{A.5}$$

In terms of these variables, the (z_1, z_2, ζ) coordinates are iteratively realized on the $k-2$ open sets $\mathcal{U}_1, \ldots, \mathcal{U}_{k-3}$ and $\mathcal{U}_{k-1}$ as follows:

$$\begin{aligned} z_1 &= u_{2j-1}^j w_{2j-1}^{j-1} = u_{2j}^j v_{2j} w_{2j}^j, \\ z_2 &= u_{2j-1}^{j-1} v_{2j-1} w_{2j-1}^{j-1} = u_{2j}^j w_{2j}^{j-1}, \\ \zeta &= u_{2j-1}w_{2j-1} = u_{2j}w_{2j}, \end{aligned} \tag{A.6}$$

or equivalently by substituting the expressions of ζ in terms of u_js and v_js

$$z_1 = u_{2j-1}\zeta^{j-1} = v_{2j}\zeta^j, \qquad z_2 = \zeta^{j-1}v_{2j-1} = u_{2j}\zeta^{j-1}. \tag{A.7}$$

Note in passing that using the relation $w_i v_{i+1} = 1$ and the identity $v_{i+1}\zeta = u_i$ following from the third relation of (A.6), one can show that on the $\mathcal{U}_{2j-1}$ patch, for instance, the equation of the singularity is given by

$$u_{2j-1}^2 + v_{2j-1}^2\zeta = \zeta^{k-1-2(j-1)}, \tag{A.8}$$

which is nothing but the defining equation of a $D_{k-(2j-1)}$ singularity. On the remaining other two open sets $\mathcal{U}_{k-2}$ and $\mathcal{U}_k$, we have $\zeta = u_{k-2}v_{k-2}w_{k-2} = u_k w_k$ and the following projections for z_2:

$$z_2 = \zeta^{\frac{k-4}{2}}u_{k-2}v_{k-2} = u_k\zeta^{\frac{k-4}{2}} \tag{A.9}$$

for k even integer and

$$z_2 = v_{k-2}\zeta^{[\frac{k}{2}]-1} = u_k v_k \zeta^{[\frac{k}{2}]-1} \tag{A.10}$$

for k odd. Note that in terms of the $\{(u_i, v_i, w_i)\}$ system of variables of the $\mathcal{U}_i$ patches, the first $k-2$ intersecting $\mathcal{C}_i$ curves are described by the following equations:

$$\begin{aligned} \mathcal{C}_i &= \{w_i v_{i+1} = 1, u_i = v_i = 0, u_{i+1} = w_{i+1} = 0\}, \\ \mathcal{C}_{i-1} \cap \mathcal{C}_i &= \{u_i = v_i = w_i = 0\}, \qquad 1 \leqslant k-2, \end{aligned} \tag{A.11}$$

while $\mathcal{C}_{k-1}$ and $\mathcal{C}_k$ are given by

$$\begin{aligned}
\mathcal{C}_{k-1} &= \{u_{k-2}v_{k-1} = 1, v_{k-2} = w_{k-2} - 1 = 0\}, \\
\mathcal{C}_k &= \{u_{k-2}v_k = 1, v_{k-2} = w_{k-2} + 1 = 0\}, \\
\mathcal{C}_{k-2} \cap \mathcal{C}_{k-1} &= \{u_{n-2} = v_{n-2} = w_{n-2} - 1 = 0\}, \\
\mathcal{C}_{k-2} \cap \mathcal{C}_k &= \{u_{n-2} = v_{n-2} = w_{n-2} + 1 = 0\}.
\end{aligned} \tag{A.12}$$

On the blown-up D_k geometry, the kN D3-branes are partitioned into k subsets of N_i D3-branes wrapping the $\mathcal{C}_i$ curves, and the initial gauge group $U(kN)$ breaks down to $\left[\otimes_{i=1}^k U(N_i)\right]$. Using (A.7), one can write down the pp geometry for the Penrose limit of type IIB superstring theory on the blown-up D_k singularity. The pp-wave metrics on the $\mathcal{U}_{2j-1}$ and $\mathcal{U}_{2j}$ patches one gets are derived from (3.1) by substituting z_1 and z_2 as in (A.3) and the Abelian differentials

$$\begin{aligned}
\mathrm{d}z_1 &= \zeta^{j-1}\mathrm{d}u_{2j-1} + (j-1)\zeta^{j-2}u_{2j-1}\,\mathrm{d}\zeta, \\
\mathrm{d}z_2 &= \zeta^{j-1}\mathrm{d}v_{2j-1} + (j-1)\zeta^{j-2}v_{2j-1}\,\mathrm{d}\zeta
\end{aligned} \tag{A.13}$$

for $\mathcal{U}_{2j-1}$ and

$$\mathrm{d}z_1 = \zeta^j\,\mathrm{d}v_{2j} + j\zeta^{j-1}v_{2j}\,\mathrm{d}\zeta, \qquad \mathrm{d}z_2 = \zeta^j\,\mathrm{d}u_{2j} + j\zeta^{j-1}u_{2j}\,\mathrm{d}\zeta \tag{A.14}$$

on the $\mathcal{U}_{2j}$ patch. In this way, the pp-wave metric on $\mathcal{U}_{2j}$ is given by

$$\begin{aligned}
\mathrm{d}s^2|_{D_k,\mathcal{U}_{2j}} = {} & -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |\zeta^{j-1}|^2(|v_{2j}\zeta|^2 + |u_{2j}|^2) + |z_2|^2)(\mathrm{d}x^+)^2 \\
& + |\zeta^j|^2(|\mathrm{d}u_{2j}|^2 + |\mathrm{d}v_{2j}|^2) + j^2|\zeta^{j-1}|^2(|v_{2j}|^2 + |u_{2j}|^2)|\mathrm{d}\zeta|^2 \\
& + j|\zeta^{j-1}|^2(\zeta\overline{u}_{2j}\,\mathrm{d}u_{2j}\,\mathrm{d}\overline{\zeta}_{2j} + \overline{\zeta}u_{2j}\,\mathrm{d}\zeta_{2j}\,\mathrm{d}\overline{u}_{2j}) \\
& + j|\zeta^{j-1}|^2(\zeta\overline{v}_{2j}\,\mathrm{d}v_{2j}d\overline{\zeta}_{2j} + \overline{\zeta}v_{2j}\,\mathrm{d}\zeta_{2j}\,\mathrm{d}\overline{v}_{2j}).
\end{aligned} \tag{A.15}$$

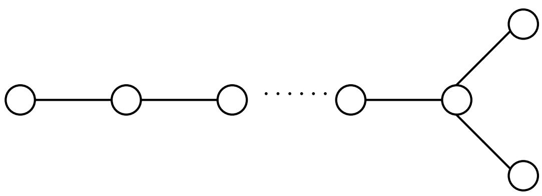
A quite similar result applies to the patch $\mathcal{U}_{2j-1}$.

Mirror D_k pp-wave geometry. Mimicking the situation for A_k, the mirror of the blow-up of the D_k singularity is obtained by introducing a system of $k+4$ complex variables τ_i satisfying the following k holomorphic constraint equations:

$$\prod_{i=1}^k \tau_i^{\ell_i^{(a)}} = 1, \tag{A.16}$$

in addition to two extra constraints. $\ell_i^{(a)}$s appearing in these equations are integers and are basically given by the Cartan matrix of the D_k finite Lie algebra. These constraint equations resemble (3.19) concerning the case A_{k-1} and are naturally solved in terms of monomials of three independent complex variables η, ξ and σ in one-to-one correspondence with the nodes of the D_k Dynkin diagram

D_k :



(A.17)

To get the corresponding pp-wave geometry, one may follow the same steps we took in the case of A_{k-1}. However, since the Dynkin diagram of D_k has a trivalent vertex exactly like its affine extension, one must resolve to methods regarding this kind of geometry. This so-called trivalent geometry is conveniently formulated in terms of elliptic fibrations over the complex plane making the study of the D_k pp-wave mirror geometry and its affine extension feasible. Details are provided below.

A.1.2. E_6 pp-wave geometry. Using the equation defining the E_6 singularity (2.25), we can eliminate variable z_1 in terms of z_2 and ζ:

$$z_1 = \sqrt{\zeta^4 + z_2^3}. \tag{A.18}$$

The differential $\mathrm{d}z_1$ may subsequently be expressed as a function of $\mathrm{d}z_2$ and $\mathrm{d}\zeta$:

$$\mathrm{d}z_1 = \frac{1}{2\sqrt{\zeta^4 + z_2^3}}\left(3z_2^2\mathrm{d}z_2 + 4\zeta^3\mathrm{d}\zeta\right). \tag{A.19}$$

The metric of a pp-wave background near the orbifold point with E_6 singularity then reads

$$\begin{aligned} \mathrm{d}s^2|_{E_6} = & -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2\left(\mathbf{x}^2 + \left|\zeta^4 + z_2^3\right| + |z_2|^2\right)(\mathrm{d}x^+)^2 \\ & + \left(1 + \frac{9|z_2|^4}{2\left|\zeta^4 + z_2^3\right|}\right)|\mathrm{d}z_2|^2 + \frac{4|\zeta|^6}{\left|\zeta^4 + z_2^3\right|}|\mathrm{d}\zeta|^2 \\ & + \frac{6}{\left|\zeta^4 + z_2^3\right|}\left(z_2^2\bar{\zeta}^3\,\mathrm{d}z_2\,\mathrm{d}\bar{\zeta} + \bar{z}_2^2\zeta^3\,\mathrm{d}\bar{z}_2\,\mathrm{d}\zeta\right). \end{aligned} \tag{A.20}$$

This is a degenerate metric at the orbifold point $\zeta = z_2 = 0$, but as before, such a degeneracy may be lifted by resolving the singularity either by blowing it up or by complex deformations. In the blow-up of the E_6 singularity, one introduces a topological basis of five open sets $\mathcal{U}_i$ parameterized by the holomorphic coordinates (u_i, v_i, w_i) and glued together as

$$v_1u_2 = w_1u_3 = w_3u_5 = w_2v_3 = w_4v_5 = 1. \tag{A.21}$$

On these open sets, $\mathcal{U}_i$, the relations for (z_1, z_2, ζ) read

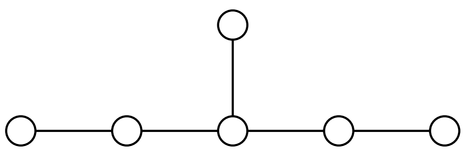
$$\begin{aligned} z_1 &= u_1v_1 = u_2v_2^6w_2 = u_3v_3^4w_3^6 = u_4v_4^2w_4^4 = u_5w_5^2, \\ z_2 &= v_1 = v_2^2w_2 = v_3^3w_3^3 = v_4^2w_4^3 = v_5w_5^2, \\ \zeta &= v_1w_1 = v_2^3w_2 = v_3^3w_3^3 = v_4^4w_4^2 = w_5, \end{aligned} \tag{A.22}$$

together with

$$\begin{aligned} u_1 &= -\sqrt{v_1 + v_1^2w_1^4}, \qquad u_2 = -\sqrt{v_2 + w_2^2}, \qquad u_3 = -\sqrt{1 + v_3}, \\ u_4 &= -\sqrt{1 + v_4^2w_4^2}, \qquad u_5 = -\sqrt{1 + v_5^3w_5^2}. \end{aligned} \tag{A.23}$$

From these relations, one can compute the explicit expressions of the $\mathrm{d}z_1, \mathrm{d}z_2$ and $\mathrm{d}\zeta$ differentials in terms of $\mathrm{d}v_i$ and $\mathrm{d}w_i$. Indeed, putting back these into (A.20), one gets the E_6 pp-wave metric on the $\mathcal{U}_i$ open sets. The mirror geometry of the E_6 singularity may also be derived by following the same lines we have presented above, now based on the Dynkin diagram

E_6:



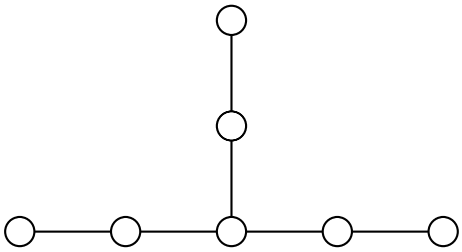
(A.24)

As for D_k, it is more convenient to use here the elliptic fibrations over the complex plane.

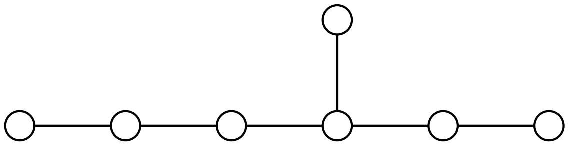
A.2. *Affine* $\widehat{\mathrm{ADE}}$ *pp waves*

Here we continue studying the Penrose limits of the $\mathrm{AdS}_5 \times S^5/\Gamma$ orbifolds with affine $\widehat{\mathrm{ADE}}$ singularities. The deformations of these singularities are represented by their Dynkin diagrams (4.19), (6.20) and

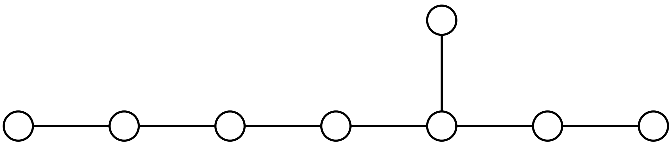
$\widehat{E}_6$:



$\widehat{E}_7$: (A.25)



$\widehat{E}_8$:



In the cases of affine $\widehat{A}_{k-1}$ and $\widehat{D}_k$, the standard defining equations of the singularities read as follows:

Affine singularity	Geometry near the orbifold point	
$\widehat{A}_{k-1}$	$z_1^2 + z_2^3 + z_2^2 = \zeta^k,$	$k \geqslant 2$
$\widehat{D}_k$	$z_1^2 + z_2^3 + z_2^2\zeta = \zeta^{k-1},$	$k \geqslant 3$

(A.26)

They differ from the ordinary ones by the extra z_2^3 terms. For the affine $\widehat{E}_6$, $\widehat{E}_7$ and $\widehat{E}_8$ geometries, the defining equations of singularities are conveniently described by the help of trivalent geometry based on elliptic fibrations over the complex plane. In this language [31], the mirrors of the blow-ups of these exceptional affine singularities read

Singularity	Mirror geometry
$\widehat{E}_6$	$(z_1^3+z_2^3+z_3^3+z_1z_2z_3)+av(z_1^2+z_2^2+z_3^2)$ $+bv^2(z_1+z_2+z_3)+cv^3$
$\widehat{E}_7$	$(z_1^2+z_2^4+z_3^4+az_1z_2z_3)$ $+\sum_{i=1}^4 v^i(a_iz_2^{4-i}+b_iz_3^{4-i})+\sum_{i=1}^2 c_iv^iz_1^{2-i}$
$\widehat{E}_8$	$(z_1^2+z_2^3+z_3^6+az_1z_2z_3)$ $+\sum_{i=1}^6 a_iv^iz_3^{6-i}+\sum_{i=1}^3 b_iv^iz_2^{3-i}+\sum_{i=1}^2 c_iv^iz_1^{2-i}$

(A.27)

where a_i, b_i and c_i are complex deformations of the singularities. The monomials describing the mirror geometry are associated with the nodes of the Dynkin diagrams of Lie algebras. Singular surfaces are recovered by setting $a_i = b_i = c_i = 0$ and describe elliptic K3 surfaces. Following [31], it is more interesting to think about affine $\widehat{A}_{k-1}$ and $\widehat{D}_k$ singularities of (A.26) in the same manner as in (A.25). In this elliptic parameterization, affine $\widehat{A}_{k-1}$ and $\widehat{D}_k$ singularities are realized in terms of homogeneous polynomials as follows:

Elliptic singularity	Polynomial realization
$\widehat{A}_{2n}$	$v(z_1^2+z_2^3+\zeta^6+az_1z_2\zeta)+\zeta^{2n+1}+z_1z_2^{n-1}$
$\widehat{A}_{2n-1}$	$v(z_1^2+z_2^3+\zeta^6+az_1z_2\zeta)+\zeta^{2n}+z_2^n$
$\widehat{D}_{2n}$	$(z_1^2+z_2^3+\zeta^6+az_1z_2\zeta)+v^2\zeta^{4n}$ $+v(z_1\zeta^{2n}+\zeta^{2n+3}+\zeta^4z_1z_2^{n-2}+\zeta^3z_2^n)$
$\widehat{D}_{2n-1}$	$(z_1^2+z_2^3+\zeta^6+az_1z_2\zeta)+v^2\zeta^{4n-2}$ $+v(z_1\zeta^{2n-1}+\zeta^{2n+2}+\zeta^3z_1z_2^{n-2}+\zeta^4z_2^n)$

(A.28)

These equations have in general non-Abelian discrete symmetries with commutative cyclic subgroups. The latter are generated by phases ω_i and act on the coordinates (z_1, z_2, ζ, v) as

$$(z_1, z_2, \zeta, v) \to \left(\omega_i^{r_1}z_1, \omega_i^{r_2}z_2, \omega_i^{r_3}\zeta, \omega_i^{r_4}v\right). \tag{A.29}$$

In this way, we have the following result:

Singularity	r_1	r_2	r_3	r_4	ω_i
$\widehat{A}_{2n}$	3	2	1	$5-2n$	$\exp\left(\frac{2\pi i}{2n+1}\right)$
$\widehat{A}_{2n-1}$	3	2	1	$6-2n$	$\exp\left(\frac{2\pi i}{2n}\right)$
$\widehat{D}_{2n}$	3	2	1	$3-2n$	$\exp\left(\frac{2\pi i}{4n}\right)$
$\widehat{D}_{2n-1}$	3	2	1	$4-2n$	$\exp\left(\frac{2\pi i}{4n-2}\right)$

(A.30)

Degenerate metric building of affine $\widehat{\mathrm{ADE}}$ pp-wave backgrounds follows the outline in the case of the finite A_k pp wave studied above. Resolutions of these singularities lift this degeneracy and, as noted before, this can be done either by Kähler or by complex deformations. In what follows, we will focus on the complex deformations of affine $\widehat{\mathrm{ADE}}$. They are classified in [31]; see also [32–34].

A.2.1. Mirror affine $\widehat{A}_k$ pp-wave metrics. Starting from (3.1) and using the equation defining the $\widehat{A}_k$ singularity in (A.28) for the case where k is an odd integer for instance, i.e., $k = 2n-1$, one can solve the relation

$$z_1^2 + az_1z_2\zeta + \left(z_2^3 + \zeta^6 + \frac{z_2^n + \zeta^{2n}}{v}\right) = 0 \tag{A.31}$$

and express z_1 in terms of the homogeneous variables z_2, ζ and v of $\mathbb{WP}^2_{(3,2,1)}$ as

$$2z_1 = az_2\zeta \pm \sqrt{(az_2\zeta)^2 - 4\left(z_2^3 + \zeta^6 + \frac{z_2^n + \zeta^{2n}}{v}\right)}. \tag{A.32}$$

Likewise, the differential $\mathrm{d}z_1$ reads in terms of $\mathrm{d}z_2$ and $\mathrm{d}\zeta$ as follows:

$$\mathrm{d}z_1 = -\frac{nz_2^{n-1} + v(3z_2 + az_1\zeta)}{v(2z_1 + az_2\zeta)}\,\mathrm{d}z_2 - \frac{2n\zeta^n + v(6\zeta^5 + az_1z_2)}{v(2z_1 + az_2\zeta)}\,\mathrm{d}\zeta + \frac{z_2^n + \zeta^{2n}}{v^2(2z_1 + az_2\zeta)}\,\mathrm{d}v. \tag{A.33}$$

The metric of the pp-wave background near the orbifold point with affine $\widehat{A}_k$ singularity is then given by

$$\begin{aligned}
\mathrm{d}s^2|_{\widehat{A}_k} &= -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2(\mathbf{x}^2 + |z_1|^2 + |z_2|^2)(\mathrm{d}x^+)^2 \\
&\quad + \left(1 + \frac{\left|nz_2^{n-1} + v(3z_2 + az_1\zeta)\right|^2}{|v(2z_1 + az_2\zeta)|^2}\right)|\mathrm{d}z_2|^2 + \frac{|2n\zeta^n + v(6\zeta^5 + az_1z_2)|^2}{|v(2z_1 + az_2\zeta)|^2}|\mathrm{d}\zeta|^2 \\
&\quad + \frac{\left|z_2^n + \zeta^{2n}\right|^2}{|v^2(2z_1 + az_2\zeta)|^2}|\mathrm{d}v|^2 + \frac{1}{|v(2z_1 + az_2\zeta)|^2}\big\{\big(nz_2^{n-1} + v(3z_2 + az_1\zeta)\big) \\
&\quad \times (2n\overline{\zeta}^n + \overline{v}(6\overline{\zeta}^5 + a\overline{z}_1\overline{z}_2))\,\mathrm{d}z_2\,\mathrm{d}\overline{\zeta} + \mathrm{h.c.}\big\} \\
&\quad + \frac{1}{|v(2z_1 + az_2\zeta)|^2}\left[\big(nz_2^{n-1} + v(3z_2 + az_1\zeta)\big)\frac{\overline{z}_2^n + \overline{\zeta}^{2n}}{\overline{v}}\,\mathrm{d}z_2\,\mathrm{d}\overline{v} + \mathrm{h.c.}\right] \\
&\quad + \frac{1}{|v(2z_1 + az_2\zeta)|^2}\left[\frac{z_2^n + \zeta^{2n}}{v}(2n\overline{\zeta}^n + \overline{v}(6\overline{\zeta}^5 + a\overline{z}_1\overline{z}_2))\,\mathrm{d}v\,\mathrm{d}\overline{\zeta} + \mathrm{h.c.}\right],
\end{aligned} \tag{A.34}$$

where z_1 is as in (A.32), while h.c. refers to the Hermitian conjugate. This pp-wave metric is degenerate at $z_1 = z_2 = \zeta = 0$. However, this degeneracy can be lifted by complex resolution by deforming (A.28) as follows:

$$\begin{aligned}
[\widehat{A}_k]_{\text{complex def.}}: \quad & v\big(z_1^2 + z_2^3 + \zeta^6 + az_1z_2\zeta\big) + \zeta^{k+1} + \sum_{i=1} a_i z_2^{2i-1}\zeta^{k-2i-2} \\
& + z_1\sum_{i=1} b_i z_2^{2i}\zeta^{k-2i-1} + \begin{cases} z_1 z_2^{\frac{k-2}{2}}, & k \quad \text{even} \quad z_2^{\frac{k+1}{2}}, \\ z_2^{\frac{k+1}{2}}, & k \quad \text{odd}, \end{cases}
\end{aligned} \tag{A.35}$$

where a_i and b_i are complex moduli. With this relation at hand, one can go ahead and solve z_1 as a function of z_2 and ζ as in (A.32) but this time with non-zero a_i and b_i, i.e.,

$$z_1(z_2, \zeta, a_i, b_i). \tag{A.36}$$

This extends the analysis we have developed for the case where $a_i = b_i = 0$. The relations are straightforwardly obtained and are similar to those we found before, albeit a bit lengthy, and since they are not used explicitly below, they are omitted. In what follows, we give a general procedure to deal with the complex deformed affine geometries.

Before coming to the other kinds of pp-wave geometries, note that under the complex deformation the degeneracy of the orbifold point with an affine $\widehat{A}_{k+1}$ singularity, embodied by the monomial ζ^{k+1}, is now lifted. The $(k+1)\times(k+1)$ matrix Π of the characters associated with the $\{\zeta^{k+1}, \tau_i, 0 \leqslant i \leqslant k-1\}$ monomials of the $\widehat{A}_{k+1}$ geometry reads

$$\Pi = \mathrm{diag}(1, \omega^2, \omega^4, \ldots, \omega^{2k-4}, \omega^{2k-2}), \tag{A.37}$$

where $\omega = \exp\left(\frac{2\pi i}{k+1}\right)$. These characters may be derived from the properties of the deformation monomials τ_i, which read for k even integer, $k = 2n$, as

$$\begin{aligned} \tau_{2i} &= z_2^i \zeta^{2n-2i+1}, \qquad && i = 0, \ldots, n, \\ \tau_{2i+1} &= z_1 z_2^{i-1} \zeta^{2n-2i}, \qquad && i = 1, \ldots, n \end{aligned} \tag{A.38}$$

and for $k = 2n + 1$ as

$$\begin{aligned} \tau_{2i} &= z_2^i \zeta^{2n-2i+2}, \qquad && i = 0, \ldots, n+1, \\ \tau_{2i+1} &= z_1 z_2^{i-1} \zeta^{2n-2i+1}, \qquad && i = 1, \ldots, n. \end{aligned} \tag{A.39}$$

These monomials satisfy equations of the type (3.19) and are related to each other as

$$\tau_{2i} = \frac{\zeta^2}{z_2}\tau_{2i+2}, \qquad \tau_{2i+1} = \frac{\zeta^2}{z_2}\tau_{2i+3}, \qquad \tau_{2i} = \frac{z_2\zeta}{z_1}\tau_{2i+1}. \tag{A.40}$$

In group-theory language, this link is described by an automorphism generated by a $(k+1) \times (k+1)$ matrix Q which in the present case reads

$$Q = \omega^2 \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & 0 & . & . & & \\ & & . & . & . & \\ & & & . & 0 & 1 \\ 1 & & & & 0 & 0 \end{pmatrix}. \tag{A.41}$$

As the ring of monomials form is closed[8], the Π and Q matrices obviously obey $\Pi^{k+1} = Q^{k+1} = \mathrm{Id}$, the $(k+1) \times (k+1)$ identity matrix. For the example of affine $\widehat{A}_6$ geometry, the Π and Q matrices read

$$\Pi(\widehat{A}_6) = \begin{pmatrix} 1 & & & & & & \\ & e^{i\frac{2\pi}{7}} & & & & & \\ & & e^{i\frac{4\pi}{7}} & & & & \\ & & & e^{i\frac{6\pi}{7}} & & & \\ & & & & e^{i\frac{8\pi}{7}} & & \\ & & & & & e^{i\frac{10\pi}{7}} & \\ & & & & & & e^{i\frac{12\pi}{7}} \end{pmatrix} \tag{A.42}$$

and

$$Q(\widehat{A}_6) = e^{i\frac{4\pi}{7}} \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ 1 & & & & & & 0 \end{pmatrix} \tag{A.43}$$

We will address these matrices again below.

[8] In the ordinary $SU(k)$ geometry, the corresponding quiver diagram is not closed and so the periodicity condition $Q^{k+1} = \mathrm{Id}$ of affine case is replaced by a nilpotency relation namely $Q^{k+1} = 0$.

A.2.2. Affine $\widehat{D}_k$ pp-wave metrics. In the case where the orbifold point has an affine $\widehat{D}_k$ singularity, one may also determine the metric of the resulting pp-wave background as follows. Since z_1 variable in (A.28) is at most quadratic, it is more convenient to put the defining equations for the affine $\widehat{D}_k$ singularity as follows:

$$z_1^2 + 2z_1 f + g = 0, \tag{A.44}$$

where the holomorphic complex functions f and g are given by

$$\begin{aligned} f(z_2,\zeta,v) &= \frac{a}{2} z_2\zeta + v\big(\zeta^k + \zeta^{4-\epsilon} z_2^{-2+\frac{k+\epsilon}{2}}\big),\\ g(z_2,\zeta,v) &= z_2^3 + \zeta^6 + v\big(\zeta^{k+3} + \zeta^{3+\epsilon} z_2^{\frac{k-\epsilon}{2}}\big). \end{aligned} \tag{A.45}$$

Using (A.44), z_1 can be solved in terms of the other variables and the solutions are as usual given by

$$z_1 = f \pm \sqrt{f^2 - g}. \tag{A.46}$$

In this way, one can then express $\mathrm{d}z_1$ in terms of $\mathrm{d}z_2$, $\mathrm{d}v$ and $\mathrm{d}\zeta$ (the holomorphic differentials of the homogeneous variables z_2, v and ζ) as follows:

$$\begin{aligned} \mathrm{d}z_1 &= \left(1 \pm \frac{f}{\sqrt{f^2-g}}\right)\mathrm{d}f \mp \frac{1}{2\sqrt{f^2-g}}\,\mathrm{d}g,\\ \mathrm{d}\chi &= \partial_{z_2}\chi\,\mathrm{d}z_2 + \partial_\zeta\chi\,\mathrm{d}\zeta + \partial_v\chi\,\mathrm{d}v, \end{aligned} \tag{A.47}$$

where χ stands for the functions f and g. The metric of pp-wave background near the orbifold point with affine $\widehat{D}_k$ singularity preserving 16 supercharges is then given by

$$\begin{aligned} \mathrm{d}s^2|_{\widehat{D}_k} = {} & -4\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\mathbf{x}^2 - \mu^2\big(\mathbf{x}^2 + \big|f \pm \sqrt{f^2-g}\big|^2 + |z_2|^2\big)(\mathrm{d}x^+)^2\\ & + \left[1 + \left|\left(1 \pm \frac{f}{\sqrt{f^2-g}}\right)\partial_{z_2} f \mp \frac{1}{2\sqrt{f^2-4g}}\partial_{z_2} g\right|^2\right]|\mathrm{d}z_2|^2\\ & + \left|\left(1 \pm \frac{f}{\sqrt{f^2-g}}\right)\partial_\zeta f \mp \frac{1}{2\sqrt{f^2-g}}\partial_\zeta g\right|^2 |\mathrm{d}\zeta|^2\\ & + \left|\left(1 \pm \frac{f}{\sqrt{f^2-g}}\right)\partial_v f \mp \frac{1}{2\sqrt{f^2-g}}\partial_v g\right|^2 |\mathrm{d}v|^2\\ & + \left[\left(1 \pm \frac{f}{\sqrt{f^2-g}}\right)\partial_{z_2} f \mp \frac{1}{2\sqrt{f^2-g}}\partial_{z_2} g\right]\\ & \times \left[\overline{\left(1 \pm \frac{f}{\sqrt{f^2-g}}\right)\partial_\zeta f \mp \frac{1}{2\sqrt{f^2-g}}\partial_\zeta g}\right]\mathrm{d}z_2\,\mathrm{d}\overline{\zeta} + \text{h.c.}\\ & + \left[\left(1 \pm \frac{f}{\sqrt{f^2-4g}}\right)\partial_{z_2} f \mp \frac{1}{2\sqrt{f^2-g}}\partial_{z_2} g\right]\\ & \times \left[\overline{\left(1 \pm \frac{f}{\sqrt{f^2-4g}}\right)\partial_v f \mp \frac{1}{2\sqrt{f^2-g}}\partial_v g}\right]\mathrm{d}z_2\,\mathrm{d}\overline{v} + \text{h.c.} \end{aligned}$$

$$
\begin{aligned}
&+\left[\left(1\pm\frac{f}{\sqrt{f^2-g}}\right)\partial_v f \mp \frac{1}{2\sqrt{f^2-g}}\partial_v g\right]\\
&\times\overline{\left[\left(1\pm\frac{f}{\sqrt{f^2-g}}\right)\partial_\zeta f \mp \frac{1}{2\sqrt{f^2-g}}\partial_\zeta g\right]}\,\mathrm{d}v\,\mathrm{d}\overline{\zeta}+\text{h.c.}
\end{aligned}
\tag{A.48}
$$

As expected, this metric is degenerate at $z_2 = \zeta = 0$. Its resolution can be obtained by performing deformations. Like for the $\widehat{A}_k$ case, complex deformations transform (A.45) to

$$
\begin{aligned}
f(z_2,\zeta,v) &= \frac{a}{2}z_2\zeta + v\big(b_1\zeta^k + c_1\zeta^{4-\epsilon}z_2^{-2+\frac{k+\epsilon}{2}}\big),\\
g(z_2,\zeta,v) &= z_2^3 + \zeta^6 + v\big(b_2\zeta^{k+3} + c_2\zeta^{3+\epsilon}z_2^{\frac{k-\epsilon}{2}}\big) + v^2\sum_{i=1}^{k-1}a_i z_2^{i-1}\zeta^{2(k+1-i)}.
\end{aligned}
\tag{A.49}
$$

In this case, the matrices Π and Q can be obtained from the properties of the deformation monomials τ_i of the mirror singularity of $\widehat{D}_k$. Since these τ_is satisfy the mirror geometry constraint equations,

$$
\prod_i \tau_i^{\ell_i^{(a)}} = 1, \tag{A.50}
$$

with $\ell_i^{(a)}$ essentially be given by minus the Cartan matrix of $\widehat{D}_k$. The matrices Π and Q read

$$
\Pi(\widehat{D}_k) = \mathrm{diag}(\alpha_1,\alpha_2,1,\omega,\omega^2,\ldots,\omega^{k-4},\alpha_k,\alpha_{k+1}) \tag{A.51}
$$

and

$$
Q(\widehat{D}_k) = \begin{pmatrix}
0 & 0 & \alpha_2 & & & & & & & \\
 & 0 & \alpha_1 & & & & & & & \\
 & & 0 & \omega & & & & & & \\
 & & & 0 & \omega & & & & & \\
 & & & & \cdot & \cdot & & & & \\
 & & & & & \cdot & \cdot & & & \\
 & & & & & & 0 & \omega & & \\
 & & & & & & & 0 & \alpha_{k+1} & \alpha_k \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{pmatrix}. \tag{A.52}
$$

The phases α_i appearing in these relations are constrained as

$$
\alpha_q^2 = 1, \qquad \alpha_1\alpha_2 = 1, \qquad \alpha_k\alpha_{k+1} = 1, \qquad \omega^{k-4} = 1, \tag{A.53}
$$

together with either $\alpha_1 = \alpha_k^*$ and $\alpha_{k+1} = \alpha_2^*$, or $\alpha_1 = \alpha_{k+1}^*$ and $\alpha_k = \alpha_2^*$.

A.2.3. Example. For the example of $\widehat{D}_6$ pp-wave geometries, the Π and Q matrices read as follows:

$$
\Pi(\widehat{D}_6) = \begin{pmatrix}
\alpha_1 & & & & & & \\
 & \alpha_2 & & & & & \\
 & & 1 & & & & \\
 & & & -1 & & & \\
 & & & & 1 & & \\
 & & & & & \alpha_6 & \\
 & & & & & & \alpha_7
\end{pmatrix} \tag{A.54}
$$

and

$$Q(\widehat{D}_6) = \begin{pmatrix} 0 & 0 & \alpha_2 & & & & \\ & 0 & \alpha_1 & & & & \\ & & 0 & -1 & & & \\ & & & 0 & -1 & & \\ & & & & 0 & \alpha_7 & \alpha_6 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}. \tag{A.55}$$

Using the same approach as in (A.44), and following the same lines we have used for the case of affine $\widehat{D}_k$, one can also work out the pp-wave geometries and the matrices Π and Q for the exceptional Lie algebras.

A.3. $\mathcal{N} = 2$ *affine* $\widehat{\mathrm{DE}}$ *CFT*$_4$

A.3.1. $\mathcal{N} = 2$ *affine* $\widehat{D}_k$ *CFT*$_4$. These $\mathcal{N} = 2$ CFT$_4$ models are obtained by solving (4.10) for affine $\widehat{D}_k$ singularity (with $k > 4$). Indeed, we have the following:

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ \vdots \\ N_{k-2} \\ N_{k-1} \\ N_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.56}$$

which is equivalent to

$$2N_1 - N_3 = 0, \qquad 2N_2 - N_3 = 0, \tag{A.57}$$

$$-N_1 - N_2 + 2N_3 - N_4 = 0, \tag{A.58}$$

$$-N_{i-1} + 2N_i - N_{i+1} = 0, \qquad 3 \leqslant i \leqslant k-3, \tag{A.59}$$

$$-N_{k-3} + 2N_{k-2} - N_{k-1} - N_k = 0, \tag{A.60}$$

$$-N_{k-2} + 2N_{k-1} = 0, \qquad -N_{k-2} + 2N_k = 0. \tag{A.61}$$

These equations are easily solved. Indeed, (A.57) tells us that $N_1 = N_2 = N$ and $N_3 = 2N$, while (A.58) and (A.59) reveal that $N_3 = N_4 = \cdots = N_{k-2} = 2N$. The last two relations require that $N_{k-1} = N_k = N$. Therefore, the resulting gauge group, G, of the $\mathcal{N} = 2$ SCFT$_4$ reads

$$G_{\widehat{D}_k} = U(N)^4 \otimes U(2N)^{k-3}. \tag{A.62}$$

Bi-fundamental matter is engineered by the links of the affine $\widehat{D}_k$ Dynkin diagram as shown in figure (A.24). The scalar matter content of this quiver theory is

$$\begin{pmatrix} Q_{1,\bar{3}} \\ \widetilde{Q}_{1,\bar{3}} \end{pmatrix}, \begin{pmatrix} Q_{2,\bar{3}} \\ \widetilde{Q}_{2,\bar{3}} \end{pmatrix}, \begin{pmatrix} Q_{3,\bar{4}} \\ \widetilde{Q}_{3,\bar{4}} \end{pmatrix}, \ldots, \begin{pmatrix} Q_{k-3,\overline{(k-2)}} \\ \widetilde{Q}_{k-3,\overline{(k-2)}} \end{pmatrix}, \begin{pmatrix} Q_{k-2,\overline{(k-1)}} \\ \widetilde{Q}_{k-2,\overline{(k-1)}} \end{pmatrix}, \begin{pmatrix} Q_{k-2,\bar{k}} \\ \widetilde{Q}_{k-2,\bar{k}} \end{pmatrix}. \tag{A.63}$$

Like in the previous example, the $Q_{i,\bar{j}}$ and $\widetilde{Q}_{i,\bar{j}}$ fields are just complex moduli of the deformation of affine $\widehat{D}_k$ singularity of the pp-wave geometry.

A.3.2. $\mathcal{N} = 2$ affine $\widehat{E}_s$ CFT$_4$. Similarly, one can show that the gauge symmetries of the remaining $\mathcal{N} = 2$ CFT$_4$ models associated with $\widehat{E}_s$ ($s = 6, 7, 8$) orbifolds are given by

$$\begin{aligned} G_{\widehat{E}_6} &= U(N)^3 \otimes U(2N)^3 \otimes U(3N), \\ G_{\widehat{E}_7} &= U(N)^2 \otimes U(2N)^3 \otimes U(3N)^2 \otimes U(4N), \\ G_{\widehat{E}_8} &= U(N) \otimes U(2N)^2 \otimes U(3N)^2 \otimes U(4N)^2 \otimes U(5N) \otimes U(6N). \end{aligned} \tag{A.64}$$

Bi-fundamental matter is associated with the links of the affine $\widehat{E}_s$ Dynkin diagrams (A.25).

Appendix B. ADE CFT$_4$s and toric geometry

B.1. Interpretation of conformal invariance in toric geometry

The vanishing condition for the beta function dictated by the requirement of conformal invariance of the $\mathcal{N} = 2$ quiver gauge theory translates into a nice Lie algebraic condition [35, 36]. This condition appears in the standard classification theorem of Kac–Moody algebras. For the special class of $\mathcal{N} = 2$ affine $\widehat{A}_k$ CFT$_4$ with gauge group $G = \left[\otimes_{i=0}^{k} U(N_i)\right]$, the above-mentioned condition reads

$$N_{i-1} - 2N_i + N_{i+1} = 0, \qquad 1 \leqslant i \leqslant k, \tag{B.1}$$

where N_i is a positive integer for all i. For affine $\widehat{\mathrm{DE}}$, the equations are slightly more complicated. In the case of $\widehat{D}_k$, for example, we still not only have conditions of the type (B.1), but also

$$N_{k-3} - 2N_{k-2} + N_{k-1} + N_k = 0. \tag{B.2}$$

According to the classification theorem of Lie algebras, the integers N_j are proportional to the Dynkin weights w_i, that is, $N_i = N w_i$, where N is a fixed positive integer. To indicate the ideas, we will focus on affine $\widehat{A}_k$ CFT$_4$s and give the results for the general case. To that purpose note that the above relations have a remarkable interpretation in toric geometry [31–34]. They may be thought of as a toric geometry relation of the form

$$\sum_j \ell_j^{(a)} \mathbf{v}_j = 0, \tag{B.3}$$

where the $\mathbf{v}_j$ vertices are given by

$$\mathbf{v}_j = (1, N_j, n_j). \tag{B.4}$$

As before, N_js are gauge-group orders, while n_js are some integers with no major interest here and will therefore be ignored. $\ell_j^{(a)}$s are given by minus the Cartan matrix, i.e., $\ell_j^{(a)} = \delta_{j+1}^a - 2\delta_j^a + \delta_{j-1}^a$. $\ell_j^{(a)}$s are integers satisfying the following toric geometry property:

$$\sum_j \ell_j^{(a)} = 0. \tag{B.5}$$

This constraint, which defines the local Calabi–Yau condition in toric geometry, is already contained in the toric equation (B.3) as the projection on the first component of the vertices $\mathbf{v}$. In toric geometry, (B.3) gives the links between the vertices. For the resolved affine $\widehat{A}_k$ singularity, for instance, a given vertex $\mathbf{v}_a$ is schematically represented by the usual $\widehat{A}_k$ Dynkin diagram

$$\cdots \bullet_{\mathbf{v}_{a-1}} \longleftrightarrow \bullet_{\mathbf{v}_a} \longleftrightarrow \bullet_{\mathbf{v}_{a+1}} \cdot \tag{B.6}$$

Observe in passing that in this $\widehat{A}_k$ case, the non-zero $\ell^{(a)}$ describing the links between the $\mathbf{v}_j$ vertices take the form $(1, -2, 1)$. The $\mathcal{N} = 2$ conformal invariance of affine $\widehat{A}_k$ CFT$_4$ is thereby encoded in the affine $\widehat{A}_k$ Dynkin diagram. The total $\sum_j p_j^a$ should vanish and the same for $\sum_j p_j^a \mathbf{v}_j$ which include $\sum_j p_j^a N_j$ as a projection on the second vector basis.

This analysis may be extended to the case of affine $\widehat{\mathrm{DE}}$ CFT, but here one should also allow vertices with more than two links. This kind of toric geometry representation has been studied in detail in [31, 32]. The geometry is given by the following typical vertex:

$$\begin{array}{ccc} 1 \searrow & & \swarrow 1 \\ & \bullet \; -2 & \\ 1 \nearrow & & \nwarrow -1 \end{array} \tag{B.7}$$

with a vector charge

$$(1, -2, 1, 1, -1) \tag{B.8}$$

satisfying the local Calabi–Yau condition (B.5). In terms of this vertex, the conformal condition may be thought of as a particular situation of the following general relation:

$$N_{i-1} - 2N_i + N_{i+1} + (J_i - L_i) = 0, \tag{B.9}$$

which can also be rewritten as

$$K_{ij} N_j = J_i - L_i. \tag{B.10}$$

Before going ahead, we note the following two features regarding the above relation: (i) as far as supersymmetric affine $\widehat{\mathrm{DE}}$ CFT$_4$s are concerned, one can interpret the constraint equations required by conformal invariance as toric geometry equations involving (B.7), (B.8) as for the affine $\widehat{A}_k$ CFT$_4$s. In the case of affine $\widehat{D}_k$ CFT$_4$s, the corresponding constraint equations, in particular (A.58) and (A.60) associated with the two trivalent vertices $\mathbf{v}_3$ and $\mathbf{v}_{k-2}$ of the $\widehat{D}_k$ Dynkin diagram

$$-N_1 - N_2 + 2N_3 - N_4 = 0, \qquad -N_{k-3} + 2N_{k-2} - N_{k-1} - N_k = 0, \tag{B.11}$$

merely correspond to a special configuration of (B.9) where we have

$$\begin{aligned} J_1 &= N_1, \qquad & J_k &= N_k, \\ L_1 &= L_k = 0, \qquad & 1 &\leqslant i \leqslant k. \end{aligned} \tag{B.12}$$

For the other bivalent vertices, we simply have $J_i = L_i = 0$ for $2 \leqslant i \leqslant k-1$.

In fact, this picture is a particular case of a more general representation extending beyond the $\widehat{\mathrm{DE}}$ Dynkin diagrams as it concerns the Dynkin diagrams of general Lie algebras. In the general case, the numbers J_i and L_i are interpreted as associated with fundamental matter and so one ends up with a larger class of $\mathcal{N} = 2$ gauge theories with fundamental matter. For example, the affine $\widehat{D}_k$ CFT$_4$ quiver gauge theory with gauge group (A.62) can, under the assumption that the YM gauge couplings g_1 and g_k associated with the first and kth nodes of the $\widehat{D}_k$ Dynkin diagram tend to zero, be thought of as an ordinary A_{k-2} CFT$_4$ quiver gauge theory with group symmetry $G_\mathrm{g} \otimes G_\mathrm{f}$ given by

$$G_\mathrm{g} = U(N)^2 \otimes U(2N)^{k-3}, \qquad G_\mathrm{f} = U(N)^2. \tag{B.13}$$

It follows that (B.10) actually describes a more general class of $\mathcal{N} = 2$ conformal quiver gauge theories. The number J_i is now just the number of fundamental matter one engineers on the nodes based on Dynkin diagrams of finite-dimensional Lie algebras. For conformal field theories with both J_i and L_i fundamental matter, we refer to [35, 36]. In what follows, we give the results for the ordinary Lie algebras.

B.2. $\mathcal{N} = 2$ D_k CFT$_4$

To get the category of $\mathcal{N} = 2D_k$ CFT$_4$ based on finite D_k Dynkin diagrams, one may mimic our approach to the supersymmetric A_{k-1} CFT$_4$. Taking the quiver group symmetry as $G_g \otimes G_f$ with gauge group $G_g = \left[\otimes_{i=1}^{k} U(N_i)\right]$ and flavour group $G_f = \left[\otimes_{i=1}^{k} U(M_i)\right]$, where each group factor $U(N_i) \otimes U(M_i)$ is engineered over the ith node of the finite D_k Dynkin diagram as in (B.7), (B.8), scale invariance requires the following conditions to be satisfied:

$$\begin{aligned} M_1 &= 2N_1 - N_2, \\ M_i &= 2N_i - (N_{i-1} + N_{i+1}), \qquad 2 \leqslant i \leqslant k-2, \\ M_k &= 2N_k - N_{k-2}, \end{aligned} \tag{B.14}$$

where $N_i, M_i \geqslant 1$. The group symmetry and the matter content for the D_k category of $\mathcal{N} = 2$ CFT$_4$s are collected in the following table:

(B.15)

Gauge group G_g	$\left[\otimes_{i=1}^{k} U(N_i)\right]$
Flavour group G_f	$U(2N_1 - N_2) \otimes U(2N_k - N_{k-2})$ $\otimes \left[\otimes_{i=2}^{k-2} U(2N_i - N_{i-1} - N_{i+1})\right]$
Bi-fundamental matter	$\oplus_{i=1}^{k-2} (N_i, \overline{N}_{i+1}) \oplus (N_{k-2}, \overline{N}_k)$
Fundamental matter	$\oplus_{i=1}^{k} (M_i N_i)$

One may recover other $\mathcal{N} = 2$ conformal models by considering the vanishing limits of some of the gauge-coupling constants.

B.3. $\mathcal{N} = 2E_6$ CFT$_4$

For a group symmetry $G_g \otimes G_f$, with quiver gauge group $G = \left[\otimes_{i=1}^{6} U(N_i)\right]$ and flavour group $G_f = \left[\otimes_{i=1}^{6} U(M_i)\right]$, scale invariance requires the following relations to hold:

$$\begin{aligned} &M_1 = 2N_1 - N_2, \qquad M_2 = 2N_2 - (N_3 + N_1), \\ &M_3 = 2N_3 - (N_2 + N_4 + N_6), \qquad M_4 = 2N_4 - (N_3 + N_5), \\ &M_5 = 2N_5 - N_4, \qquad M_6 = 2N_6 - N_3, \end{aligned} \tag{B.16}$$

where $N_i, M_i \geqslant 1$. The symmetries in $\mathcal{N} = 2E_6$ CFT$_4$s are as follows:

(B.17)

Gauge group G_g	$\left[\otimes_{i=1}^{6} U(N_i)\right]$
Flavour group G_f	$U(2N_1 - N_2) \otimes U(2N_2 - N_3 - N_1)$ $\otimes U(2N_3 - N_2 - N_4 - N_6) \otimes U(2N_4 - N_3 - N_5)$ $\otimes U(2N_5 - N_4) \otimes U(2N_6 - N_3)$
Bi-fundamental matter	$(N_1, \overline{N}_2) \oplus (N_2, \overline{N}_3) \oplus (N_3, \overline{N}_4)$ $\oplus (N_3, \overline{N}_6) \oplus (N_4, \overline{N}_5)$
Fundamental matter	$\oplus_{i=1, i\neq 3}^{6} (M_i N_i)$

This result is easily extended to the other exceptional groups, E_7 and E_8.

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